

Congruences between modular forms over imaginary
quadratic fields

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Outline:

Apply

congruences involving Eisenstein cohomology classes over an imaginary quadratic field F to

- (A) construct elts in Selmer groups of Hecke characters of F ,
- (B) prove modularity of residually reducible $\text{Gal}(\overline{F}/F)$ -representations (joint work in progress with K. Klosin).

Modular Forms over F :

- real analytic functions on $\mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$
- automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$
- cohomology classes in $H^i(\Gamma \backslash \mathbf{H}_3, R)$, where $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$ and R is an \mathcal{O}_F -algebra.

Setup:

- $p > 3$ prime, $p \nmid d_F, p \nmid \#\text{Cl}(F)$, fix $F \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$,
 Σ finite set of places; \mathcal{O} ring of integers in (suff. large) finite extension K of \mathbf{Q}_p , put $G_F := \text{Gal}(\overline{\mathbf{Q}}/F)$,
- $\chi : F^* \backslash \mathbf{A}_F^* \rightarrow \mathbf{C}^*$ unramified Hecke character s.t. $\chi_\infty(z) = z^2$,
- Let \mathbf{T} be a finite \mathcal{O} -algebra generated by Hecke operators $T_v, v \notin \Sigma$, acting on weight 2 ordinary cuspidal automorphic forms on $\text{GL}_2(\mathbf{A}_F)$ of some level and character,
- Call ideal $\mathbf{I}_\chi \subset \mathbf{T}$ generated by $\{T_v - (1 + (\chi \cdot \text{Nm})(\pi_v)) \mid v \notin \Sigma\}$ the **Eisenstein ideal** associated to χ .

Theorem 1.

$$\text{val}_p \#(\mathbf{T}/\mathbf{I}_\chi) \geq \text{val}_p \left(\frac{L(0, \chi)}{\Omega^2} \right).$$

Sketch of proof: use integral structure from cohomology of a symmetric space S associated to GL_2/F (connection to automorphic forms via Eichler-Shimura-Harder)

- use Harder's theory to construct cohomology class $\text{Eis } \omega_\chi \in H^1(S, K)$ with eigenvalues $1 + (\chi \cdot \text{Nm})(\pi_v)$ and integral restriction to the boundary

- bound denominator δ of $\text{Eis } \omega_\chi$ from below by L -value using analytic methods
- prove congruence of $\delta \cdot \text{Eis } \omega_\chi$ with cuspidal class in $H_c^1(S, \mathcal{O})$ by ruling out congruence with torsion in $H_c^2(S, \mathcal{O})$. \square

Theorem 2 (B, Harcos). *Let π be a regular algebraic cuspidal automorphic representation with central character ω such that $\omega^c = \omega$. Then there exists a continuous irreducible*

$$\rho_\pi : \text{Gal}(\overline{\mathbb{Q}}/F) \rightarrow \text{GL}_2(\overline{\mathbb{Q}}_p)$$

such that $L_v(s, \pi) = L_v(s, \rho_\pi)$

$$\forall v \nmid p d_F \text{cond}(\pi) \text{cond}(\pi^c).$$

Remark. *This strengthens a result of Taylor who showed $L_v(s, \pi) = L_v(s, \rho_\pi)$ for a set of v of Dirichlet density one.*

Our proof initially follows the strategy of Taylor but results of Weissauer and Laumon allow us to simplify his proof:

Sketch of proof:

- lift many quadratic twist $\pi \otimes \mu$ to cuspidal automorphic representation Π^μ of $\mathrm{GSp}_4(\mathbf{A}_{\mathbf{Q}})$ (Harris, Soudry, Taylor),

- use congruences to higher weight forms, results of Weisauer/Laumon, and theory of pseudorepresentations to get

$$\tau^\mu : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \mathrm{GL}_4(\overline{\mathbf{Q}}_p)$$

such that $L^S(s, \tau^\mu) = L^S(s, \mathrm{I}_F^{\mathbf{Q}}(\pi \otimes \mu))$,

- show existence of $\rho^\mu : G_F \rightarrow \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ such that $\tau^\mu \cong \mathrm{ind}_F^{\mathbf{Q}}(\rho^\mu)$,

- prove existence of ρ such that $\rho^\mu \oplus (\rho^\mu)^c \cong (\rho \otimes \mu) \oplus (\rho \otimes \mu)^c$,
- Let $\{\alpha_v, \beta_v\}$ inverse roots of Hecke poly of π , $\{\gamma_v, \delta_v\}$ inverse roots of Frobenius poly of ρ , get

$$\begin{aligned} & \{\gamma_v \mu(v), \delta_v \mu(v), \gamma_{v^c} \mu(v^c), \delta_{v^c} \mu(v^c)\} \\ &= \{\alpha_v \mu(v), \beta_v \mu(v), \alpha_{v^c} \mu(v^c), \beta_{v^c} \mu(v^c)\}, \end{aligned}$$

consider μ_1, μ_2 such that $\mu_1(v) = \mu_1(v^c)$ and $\mu_2(v) \neq \mu_2(v^c)$.

□

Theorem 3 (Urban). *If π is ordinary at $v \mid p$ then $\rho_\pi|_{D_v}$ is ordinary.*

(A): Construct elements in Selmer groups

(Strategy of Ribet, Wiles, Skinner-Urban,...)

For $\lambda : G_F \rightarrow \mathcal{O}^*$, a continuous p -adic Galois character, let \mathcal{O}_λ be the free rank one \mathcal{O} -module with G_F -action by λ .

Selmer group $\text{Sel}(F, \lambda) \subset H^1(G_F, \mathcal{O}_\lambda \otimes K/\mathcal{O})$,

unramified away from p

$\left\{ \begin{array}{ll} \text{ordinary at } v \mid p & \text{if } p \text{ split,} \\ \text{crystalline} & \text{if } p \text{ inert.} \end{array} \right.$

Let χ_p be the p -adic Galois character associated to χ by Weil, ϵ the p -adic cyclotomic character.

Theorem 4.

$$\text{val}_p \#(\mathbf{T}/\mathbf{I}_\chi) \leq \text{val}_p \# \text{Sel}(F, \chi_{p\epsilon}).$$

Idea of proof:

- For $\pi \equiv \text{Eis } \omega_\chi \pmod{\pi_K}$ use techniques of Wiles/Urban to construct suitable lattice for ρ_π ;
get an element of $H^1(G_F, \mathcal{O}_{\chi_{p\epsilon}} \otimes K/\mathcal{O})$,
- properties of $\rho_\pi \Rightarrow$ element of $\text{Sel}(F, \chi_{p\epsilon})$.



Remarks:

- Bloch-Kato conjecture implies that

$$\mathrm{val}_p \# \mathrm{Sel}(F, \chi_p^{-1}) = \mathrm{val}_p \left(\frac{L(0, \chi)}{\Omega^2} \right).$$

- By generalised Cassels-Tate pairing $\mathrm{Sel}(F, \chi_p \epsilon) \cong \mathrm{Sel}(F, \chi_p^{-1})$.
- Certain cases of p-primary Bloch-Kato have been proven from Main Conjecture for imaginary quadratic fields, including this critical, weight -2 case (Kato, Guo, Han, Tsuji).
- Interesting about our method is that we just work with congruences over F .

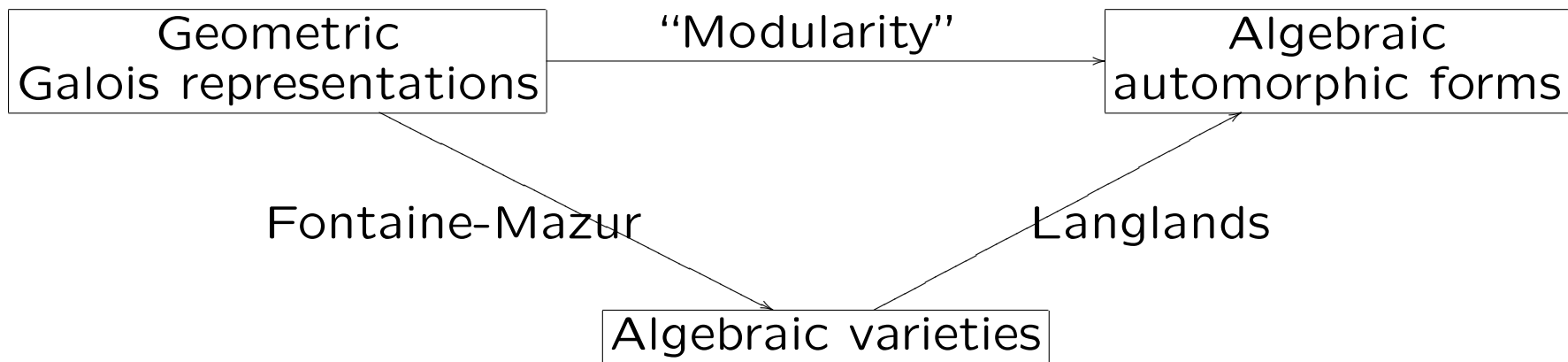
(B): Modularity of Galois representations

Let L be a totally real number field. Call an irreducible continuous representation

$$\rho : G_L \rightarrow \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$$

geometric if ρ is unramified outside a finite set of places and potentially semistable at primes dividing p .

Conjecture (Fontaine-Mazur, 1995) ρ is geometric $\Leftrightarrow \rho$ is isomorphic to a (Tate twist) of the Galois representation associated to a Hilbert modular newform.



Results: \Rightarrow is proven using Serre's Conjecture (Theorem of Khare-Wintenberger) + Modularity Lifting Theorems

Work of Wiles, Diamond, Fujiwara, Skinner-Wiles[3] (ordinary representations), Taylor, L. Berger-Breuil, Kisin, Colmez, Emerton...

Skinner-Wiles[1,2] study residually reducible representations.

Back to setup for $F = \text{imaginary quadratic}$:

- Assume in addition p splits, $(p) = \mathfrak{p}\bar{\mathfrak{p}}$, let G_Σ be the Galois group of the maximal extension of F unramified outside Σ .
- Put $\tau = \overline{\chi_p \epsilon} : G_\Sigma \rightarrow k^\times$, k residue field of \mathcal{O} ;
note $\tau|_{I_{\mathfrak{p}}} = \bar{\epsilon}^{-1}|_{I_{\mathfrak{p}}}$ and $\tau|_{I_{\bar{\mathfrak{p}}}} = \bar{\epsilon}|_{I_{\bar{\mathfrak{p}}}}$.
- Assume
 - The τ^{-1} -eigenspace of the p -part of the class group of the splitting field $F(\tau)$ of τ is trivial,
 - If $v \in \Sigma$, then either τ is ramified at v or $\tau^{-1}(\text{Frob}_v) \neq \#k_v$.

Theorem 5. *Let $\rho, \rho' : G_\Sigma \rightarrow \mathrm{GL}_2(k)$ be two Galois representations satisfying*

- $\rho, \rho' = \begin{pmatrix} 1 & * \\ & \tau \end{pmatrix},$
- ρ, ρ' have scalar centraliser.

Then $\rho' \cong \rho.$

Proof. Careful analysis of Galois structure of local and global units of $F(\tau)$. □

Fix a Galois representation

$$\rho_0 = \begin{pmatrix} 1 & * \\ 0 & \tau \end{pmatrix} : G_\Sigma \rightarrow \mathrm{GL}_2(k)$$

as in the Theorem.

Goal: study deformations of ρ_0 :

Definition. An \mathcal{O} -deformation of ρ_0 is a local complete Noetherian \mathcal{O} -algebra (A, k, \mathfrak{m}_A) and $\rho : G_\Sigma \rightarrow \mathrm{GL}_2(A)$ such that $\rho_0 = \rho \bmod \mathfrak{m}_A$.

Question: Are they (under appropriate conditions) all modular, i.e. arise from Galois representations attached to cuspforms of $\mathrm{GL}_2(\mathbf{A}_F)$?

Existence of modular deformations:

Corollary(of Theorems 1,2,3).

If $p \mid \frac{L(0,\chi)}{\Omega^2}$ then there exists an ordinary cuspform π and $\rho \cong \rho_\pi$ such that $\bar{\rho} = \rho_0$, i.e. an irreducible ordinary deformation.

Corollary. ρ_0 splits when restricted to $D_{\bar{p}}$.

Existence of a reducible deformation:

Put $\Psi = \chi_p \epsilon : G_\Sigma \rightarrow \mathcal{O}^*$.

Definition. For T a finite set of places of F let $L_\Psi(T)$ be the maximal abelian pro- p extension of $F(\Psi)$ unramified outside T such that $\text{Gal}(F(\Psi)/F)$ acts on $\text{Gal}(L_\Psi(T)/F(\Psi))$ by Ψ^{-1} .

A result of Greenberg shows:

Proposition 6.

$\text{Gal}(L_\Psi(\Sigma \setminus \{\bar{p}\})/F(\Psi))$ is \mathbf{Z}_p -torsion.

This means that there exists no reducible ordinary \mathcal{O} -deformation of ρ_0 , since an ordinary representation of the form $\begin{pmatrix} 1 & * \\ 0 & \psi \end{pmatrix}$ has to split when restricted to $I_{\bar{p}}$.

This is different from the situation over \mathbb{Q} studied by Skinner and Wiles in [SW1].

However, the result of Greenberg also tells us:

Proposition 7.

The \mathbf{Z}_p -rank of $\text{Gal}(L_\Psi(\Sigma)/F(\Psi))$ is 1.

This means that we can define a *nearly ordinary* reducible deformation of ρ_0 of the form

$$\rho^{\text{Eis}} = \begin{pmatrix} 1 & * \\ 0 & \Psi \end{pmatrix}$$

which does not split when restricted to $I_{\bar{p}}$.

Note that this deformation is not de Rham at \bar{p} .

Question:

Are there conditions under which the universal ordinary deformation ring R^{ord} is a discrete valuation ring and generated as an \mathcal{O} -algebra by traces (studied over \mathbb{Q} for non-ordinary deformations by Bellaïche, Chenevier, Calegari)?

Then this would imply an $R^{\text{ord}} = T$ theorem, i.e. prove the modularity of residually reducible Galois representations.

Thank you.

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