Arithmetic properties of similitude theta lifts from orthogonal to symplectic groups

Tobias Berger

7 January 2012

Abstract

Building on the work of Kudla and Millson we obtain a lifting of cuspidal cohomology classes for the symmetric space associated to $\mathrm{GO}(V)$ for an indefinite rational quadratic space V of even dimension to holomorphic Siegel modular forms on $\mathrm{GSp}_n(\mathbf{A})$. For $n = 2$ we prove Thom's Lemma for hyperbolic 3-space, which together with results of Kudla and Millson imply an interpretation of the Fourier coefficients of the theta lift as period integrals of the cohomology class over certain cycles and relates those over infinite geodesics to L-values of cuspforms for GL_2 over imaginary quadratic fields. This allows us to prove, for almost all primes p, the p-integrality of the lift for a particular choice of Schwartz function. We further calculate the Hecke eigenvalues (including for some "bad" places) for this choice in the case of V of signature $(3, 1)$.

1 Introduction

The theta correspondence provides an important method of transfering automorphic forms between different groups. It was Shimura who started studying the arithmeticity of theta correspondences and its applications to the arithmetic of period ratios and special L-values. Harris and Kudla extended this work and in certain cases proved rationality of the theta lift over specified number fields (see e.g. [HK92]). Prasanna proved p-integrality of the Jacquet-Langlands-Shimizu correspondence [Pra06] and the Shimura and Shintani correspondences [Pra09]. We refer the reader to [Pra08] for an overview of these and other results by Finis, Harris-Li-Skinner, Emerton, and Hida. The p-integrality of a particular theta lift also allows one to use it (via congruences to "stable" forms and their associated Galois representations) to prove cases of the Bloch-Kato conjecture of Galois representations, similar to how Eisenstein series were used by Ribet in his proof of the converse to Herbrand's theorem (see e.g. [BDSP10] and [AK10] for such an application of p-integrality in the case of the Yoshida lift). In this paper we study the p-integrality of the theta lift from orthogonal similitude groups of orthogonal spaces of signature (s, t) with $s > t$ odd to symplectic similitude groups. The case of signature $(3, 1)$ is of particular interest because of its connection to GL_2 over an imaginary quadratic field (cf. [HST93]).

In Section 3 we start by explaining how to extend the work of Kudla and Millson [KM86], [KM87], [KM88], [KM90] on cohomological theta lifts for isometry groups to the similitude case. For this we adapt the definition of the similitude theta lift for automorphic forms used by [HK92]. We obtain a lifting of cuspidal cohomology classes for the symmetric space associated to a group of orthogonal similitudes to a holomorphic Siegel modular form on $\mathrm{GSp}_n(\mathbf{A})$ of weight $\frac{1}{2}(s + t)$. The results of Kudla and Millson further imply an interpretation of its Fourier coefficients as period integrals of the cohomology class over certain cycles (see Theorem 6). Since infinite geodesics were not treated by Kudla and Millson we provide a proof for these of the so-called Thom's Lemma in the case of hyperbolic 3-space in Section 4.3. Our calculation shows that these period integrals can be expressed in terms of special L-values of cuspforms for GL_2 over imaginary quadratic fields. This could be used to to identify conditions under which the theta lift is non-vanishing. By choosing arithmetic (i.e. algebraic, rational or p-integral) Schwartz functions our result allows a lifting to arithmetic Siegel modular forms (see Section 5.2). Here we take as rational (or p-integral) structures the one coming from Betti cohomology for the orthogonal group, whilst we use the Fourier expansion for the Siegel modular form.

Different to Yoshida's work on Siegel theta lifts [Yos80, Yos84, Yos86] our similitude theta lift is defined directly on GSp_n, and not as an extension of the isometry lift using strong approximation for a particular congruence subgroup. This allows us to directly localize the action of Hecke operators. In Section 5.3 we calculate the Hecke action on our similitude theta lift, including for some "bad places" dividing the level. Previously, the only calculations of the Hecke action on symplectic theta lifts for indefinite quadratic forms with $s \neq t$ had been for signature $(2,1)$ (Waldspurger and Prasanna [Pra09]). One of our motivations for this work is to obtain a p-integral lifting of ordinary cuspforms π for GL_2 over an imaginary quadratic field to an ordinary Siegel form, which would allow one to study π by variational techniques using Hida families for GSp_2. Combined with the Jacquet-Langlands correspondence our results provide such a lifting for forms with unramified or Steinberg local components π_v for $v \mid p$. We plan to investigate this and the question of non-vanishing of the theta lift in further work.

Using the work of Funke and Millson [FM06] it should be possible to extend our results to higher weights, i.e. the case of cohomology classes of the similitude orthogonal group with non-trivial coefficients and vector-valued Siegel modular forms.

The author would like to thank Chris Skinner for his suggestion to study the arithmetic of the theta lift used in the work of Harris, Soudry and Taylor [HST93], and Jim Cogdell for pointing him towards the work of Kudla and Millson. Thanks also to Stephen Kudla and Jens Funke for helpful discussions and answering questions about their work.

2 Notation and terminology

2.1 Orthogonal Groups

Let $(V, (\ ,\ ))$ be a nondegenerate quadratic space over $\mathbf{Q}$ of even dimension m and signature (s,t) with $s > t$ and t odd. Let $\chi_V(x) = (x, (-1)^{m/2}\det V) : \mathbf{Q}^* \backslash \mathbf{A}^* \to \{\pm 1\}$ be the quadratic Hecke character associated to V. Let $H = \mathrm{GO(V)}$ denote the group of orthogonal similitudes of V and let $\lambda : H \to \mathbf{G}_m$ denote the multiplier character. Put $H_1 = \mathrm{O}(V) = \ker(\lambda)$. Write H_* for $* = \emptyset, 1$.

Let D be the symmetric space of dimension $d := st$ given by the space of negative t-planes in $V(\mathbf{R})$, i.e.

$$D = \{Z \in \mathrm{Gr}_t(V(\mathbf{R})) : (\ ,\ )|_Z < 0\}.$$

Fix a basepoint $z_0 \in D$, with corresponding stabilizer $K_\infty \cong O(s) \times O(t) \subset H_1(\mathbf{R})$ so that $D \cong H_1(\mathbf{R})/K_\infty$. Since $s \neq t$ we have $H(\mathbf{R}) = H_1(\mathbf{R})\mathbf{R}^*$ (see Table 1 in Appendix of [Rob01]). We fix an orientation on V and of z_0 and propagate this orientation of z_0 continuously to all other $z \in D$. We obtain an orientation on D, i.e. an orientation of $T_z(D) \cong \mathrm{Hom}(z, z^\perp)$ depending continously on z by demanding that the orientation of $z^\perp$ followed by that of z is the orientation of V. Let $H_*(\mathbf{R})_+$ be the group of elements of $H_*(\mathbf{R})$ whose image in $H_*^{\mathrm{ad}}(\mathbf{R})$ lies in its identity component $H_*^{\mathrm{ad}}(\mathbf{R})^+$, given by the image of $\mathrm{SO}^+(V(\mathbf{R}))$. Put $H_*(\mathbf{Q})_+ = H_*(\mathbf{Q}) \cap H_*(\mathbf{R})_+$.

For an open compact subgroup $K \subset H_*(\mathbf{A}_f)$ put

$$S_K^* = H_*(\mathbf{Q})_+ \backslash D \times H_*(\mathbf{A}_f)/K.$$

The connected components of S_K^* can be described as follows. Write

$$H_*(\mathbf{A}_f) = \coprod_j H_*(\mathbf{Q})_+ h_j K \tag{1}$$

for $h_j \in H_*(\mathbf{A}_f)$. Then

$$S_K^* = \coprod_j \Gamma_j \backslash D,$$

where Γ_j is the image in $H_*^{\mathrm{ad}}(\mathbf{R})^+$ of

$$\Gamma_j' = H_*(\mathbf{Q})_+ \cap h_j K h_j^{-1}. \tag{2}$$

Note that

$$\Omega^n(S_K^*) \cong [\Omega^n(D) \otimes C^\infty(H_*(\mathbf{A}_f))]^{H_*(\mathbf{Q})_+ \times K} \cong \bigoplus_j \Omega^n(D)^{\Gamma_j}, \tag{3}$$

where the second isomorphism is obtained by evaluation at the h_j.

Let $\mathfrak{g}$ be the Lie algebra of $H_1(\mathbf{R})$, and $\mathfrak{k}$ that of K_∞. We then have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ with $\mathfrak{p}$ the orthogonal complement of $\mathfrak{k}$ with respect to the Killing form.

2.2 Symplectic groups

For $n \geq 1$ let $G = \mathrm{GSp}(n) \subset \mathrm{GL}_{2n}$ and $G_1 = \mathrm{Sp}_n$, defined using the symplectic form represented by $\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We again write λ for the multiplier of G. For $g \in G(\mathbf{A})$ put $g_1 = g \begin{pmatrix} 1 & 0 \\ 0 & \lambda(g)^{-1} \end{pmatrix} \in G_1(\mathbf{A})$. Let $G(\mathbf{A})^+$ be the subgroup of $g \in G(\mathbf{A})$ such that $\lambda(g) \in \lambda(H(\mathbf{A}))$ and $G_+(\mathbf{R})$ the elements of $G(\mathbf{R})$ with positive multiplier.

Let $L_\infty \cong U(n)$ be the maximal compact subgroup of $G_1(\mathbf{R})$. Let $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ be the complexified Cartan decomposition of $\mathfrak{sp}_n$. Recall that $\mathbf{H}_n = \{\tau \in \mathrm{Sym}_n(\mathbf{C}) : \mathrm{Im}(\tau) > 0\} \cong G_1(\mathbf{R})/L_\infty$. We can identify $\mathfrak{p}'$ with the complex tangent space of $\mathbf{H}_n$ at the point $i1_n$, and the Harish-Chandra decomposition $\mathfrak{p}' = \mathfrak{p}_+ \oplus \mathfrak{p}_-$ gives the splitting of $\mathfrak{p}'$ into the holomorphic and anti-holomorphic tangent spaces. Put $\mathfrak{q} = \mathfrak{k}' \oplus \mathfrak{p}_-$.

For $L \subset G_1(\mathbf{A}_f)$ a compact open subgroup denote the corresponding Shimura variety by

$$M_L = G_1(\mathbf{Q}) \backslash \mathbf{H}_n \times G_1(\mathbf{A}_f)/L.$$

3 Theta lift

3.1 Isometry theta lift

Fix the nondegenerate additive character $\psi = \prod_\ell \psi_\ell$ of $\mathbf{A}$ trivial on $\mathbf{Q}$ given by $\psi_\infty(x) = \exp(2\pi i x)$ and, for every rational prime ℓ, $\psi_\ell(x) = \exp(2\pi i \mathrm{Fr}_\ell(x))$ for $x \in \mathbf{Q}_\ell$, where $\mathrm{Fr}_\ell(x)$ denotes the fractional part of x. Let $1 \leq n \leq s$. Let $\omega = \omega_\psi$ denote the usual actions of $G_1(\mathbf{A}) \times H_1(\mathbf{A})$ on the Schwartz-Bruhat space $S(V(\mathbf{A})^n)$ of $V(\mathbf{A})^n$ characterized by the following actions of $G_1(\mathbf{Q}_\ell) \times H_1(\mathbf{Q}_\ell)$ on $\varphi \in S(V(\mathbf{Q}_\ell)^n)$:

$$\omega(1,h)\varphi(x) = \varphi(h^{-1}x), \tag{4}$$

$$\omega(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix},1)\varphi(x) = \psi(\frac{1}{2}\mathrm{tr}((bx,x)))\varphi(x), \tag{5}$$

$$\omega(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix},1)\varphi(x) = \chi_V(\det a)|\det a|_\ell^{m/2}\varphi(xa), \tag{6}$$

$$\omega(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},1)\varphi(x) = \gamma\hat{\varphi}(x). \tag{7}$$

Here the Fourier transform is defined by

$$\hat{\varphi}(x) = \int_{V(\mathbf{Q}_\ell)^n} \varphi(y)\psi(\mathrm{tr}(x,y))dy$$

and γ is a certain complex number of absolute value 1. For $\ell = \infty$, the action is described by the same formulae, except that we follow [KM90] Section 3 by twisting the action in (4) by the n-th power of the spinor norm (the character of $H_1(\mathbf{R})$ which is given by $1 \otimes \det$ on $K_\infty \cong O(s) \times O(t)$).

Let $K \subset H_1(\mathbf{A}_f)$ be a compact open subgroup. Denote by $S(V(\mathbf{A})^n)^K$ the K-invariant Schwartz functions. Evaluation at z_0 induces an isomorphism

$$[S(V(\mathbf{A}^n))^K \otimes \Omega^{d-i}(D)]^{H_1(\mathbf{R})} \cong [S(V(\mathbf{A}^n))^K \otimes \Lambda^{d-i}\mathfrak{p}^*]^{K_\infty}. \tag{8}$$

For $g \in G_1(\mathbf{A}), h \in H_1(\mathbf{A}_f)$ and $\phi \in [S(V(\mathbf{A}^n))^K \otimes \Omega^{d-i}(D)]^{H_1(\mathbf{R})}$ defining a closed $d-i$-form on D let

$$\theta(g,h,\phi) = \sum_{x \in V(\mathbf{Q})^n} \omega(g,h)\phi(x). \tag{9}$$

This defines a closed $d-i$-form $\theta(g,\phi)$ on S^1_K.

Following [KM90] we define a pairing

$$H^i_c(S^1_K,\mathbf{C})\times H^{d-i}_{\mathrm{ct}}(H_1(\mathbf{R}),S(V(\mathbf{A})^n)^K)\to C^\infty(G_1(\mathbf{A}))$$

by

$$\langle[\eta],[\phi]\rangle_K(g)=\int_{S^1_K}\eta\wedge\theta(g,\phi). \tag{10}$$

To calculate this using $(\mathfrak{g},K_\infty)$-cohomology, we define (following [KM87] p.274 and [HK92] p.83) the following right-invariant Haar measure on $H_1(\mathbf{A})$: The restriction of the killing form to $\mathfrak{p}$ yields an $H_1(\mathbf{R})$-invariant metric on $D\cong H_1(\mathbf{R})/K_\infty$, and a corresponding measure $d\mu$ on D. We normalize the Haar measure $dh_{1,\infty}$ by taking $dh_{1,\infty}=d\mu dk_\infty$ where $\mathrm{vol}(K_\infty)=1$. We extend this to a Haar measure dh on $H(\mathbf{Q})_+\mathbf{R}^*\backslash H(\mathbf{A}_f)\times H_1(\mathbf{R})\mathbf{R}^*$ such that the volume of the maximal compact open of $H(\mathbf{A}_f)$ is 1. The similitude factor λ induces a map

$$H(\mathbf{Q})_+\mathbf{R}^*\backslash H(\mathbf{A}_f)\times H_1(\mathbf{R})\mathbf{R}^*\to\mathbf{Q}^*\backslash\mathbf{A}_f^*.$$

We now fix the Haar measure $d\xi$ on the compact group $\mathbf{Q}^*\backslash\mathbf{A}_f^*$ so that its volume is 1. Then there is a unique measure dh_1 on $H_1(\mathbf{Q})_+\backslash H_1(\mathbf{A})$ such that $dh=dh_1d\xi$. Note that the action of $H(\mathbf{Q})_+$ on the set $H_1(\mathbf{Q})_+\backslash H_1(\mathbf{A})$ induced by conjugation preserves this measure.

Identifying de Rham forms and Lie algebra cocycles under the isomorphisms

$$\Omega^i(S^1_K)\cong[C^\infty(H_1(\mathbf{Q})_+\backslash H_1(\mathbf{A}))\otimes\Lambda^i\mathfrak{p}^*]^{K_\infty K}$$

and (8), we obtain

$$\langle[\eta],[\phi]\rangle_K=\int_{H_1(\mathbf{Q})_+\backslash H_1(\mathbf{A})/KK_\infty}\eta(h_1)\wedge\theta(g,h_1,\phi)dh_1(\mathbf{1}_\mathfrak{p}),$$

where $\mathbf{1}_\mathfrak{p}$ is an orthonormal properly oriented basis vector for $\Lambda^d\mathfrak{p}$ with respect to the metric and orientation fixed above.

We recall the following notation from [KM90]: If $\mathfrak{m}$ is a Lie algebra, $\chi:\mathfrak{m}\to\mathbf{C}$ a homomorphism and U and $\mathfrak{m}$-module, then

$$U^{\mathfrak{m}}_\chi=\{u\in U : x.u=\chi(x)u \text{ for all } x\in\mathfrak{m}\}.$$

We denote the subspace of *holomorphic* Schwartz classes, i.e. the classes annihilated by $\mathfrak{p}_-$, by

$$H^{d-i}_{\mathrm{ct}}(H_1(\mathbf{R}),S(V(\mathbf{A})^n))^{\mathfrak{p}_-}.$$

For holomorphic automorphic forms in the adelic setting we refer the reader to [Har84] Section 2. Let χ_m denote the character of L_∞ given by $\chi_m(k)=\det(k)^{m/2}$. For $L\subset G_1(\mathbf{A}_f)$ compact open let $L_\infty L$ act on $(G_1(\mathbf{Q})\backslash G_1(\mathbf{A}))\times\mathbf{C}_{\chi_m}$ by

$$(g,v)\cdot\ell=(g\ell,\chi_m(\ell_\infty^{-1})v)$$

for $g\in G_1(\mathbf{Q})\backslash G_1(\mathbf{A})$ and $\ell=(\ell_\infty,\ell_f)\in L_\infty\times L$. The projection $G_1(\mathbf{Q})\backslash G_1(\mathbf{A})\to M_L$ identifies the quotient $G_1(\mathbf{Q})\backslash G_1(\mathbf{A})\times\mathbf{C}_{\chi_m}/L_\infty L$ with a holomorphic vector bundle $\mathcal{L}_m$ over M_L. We are going to identify global sections of $\mathcal{L}_m$ with the holomorphic automorphic forms $\mathcal{H}_{\chi_m}(G_1(\mathbf{Q})\backslash G_1(\mathbf{A})/L,\mathbf{C}_{\chi_m})$ of weight $m/2$ and level L, defined as

$$\{\phi\in C^\infty(G_1(\mathbf{Q})\backslash G_1(\mathbf{A})/L,\mathbf{C}_{\chi_m}):\phi(g\ell)=\chi_m(\ell^{-1})\phi(g),\ell\in L_\infty,g\in G_1(\mathbf{A});\mathfrak{p}_-\phi=0\}.$$

Adopted to our adelic setting, Theorem 1 of [KM90] states:

Theorem 1. *The induced pairing*

$$\langle\,,\,\rangle_K : H^i_c(S^1_K,\mathbf{C})\times H^{d-i}_{\mathrm{ct}}(H_1(\mathbf{R}),S(V(\mathbf{A})^n)^K)^{\mathfrak{q}}_{\chi_m}\to\Gamma(\mathcal{L}_m)$$

takes values in the holomorphic sections, i.e., in the space of Siegel modular forms of weight $m/2$.

3.2 Similitude theta lift

We adapt the definition of the similitude theta lift for automorphic forms in [HK92] to extend the cohomological theta lifts of Kudla and Millson.

Recall from Roberts [Rob01] the definition of the extended Weil representation for

$$R(\mathbf{A}) = \{(g,h) \in G(\mathbf{A}) \times H(\mathbf{A}) : \lambda(g) = \lambda(h)\}.$$

For $h \in H(\mathbf{A})$ and $\varphi \in S(V(\mathbf{A})^n)$, let

$$L(h)\varphi(x) = |\lambda(h)|^{-mn/4}\varphi(h^{-1}x).$$

Now $(g,h) \in R(\mathbf{A})$ acts via $\omega(g,h) := \omega(g_1)L(h)$ (see Section 2.2 for definition of g_1). The actions of $G_1(\mathbf{A})$ and $H(\mathbf{A})$ do not commute, but as in [HK92] Lemma 5.1.2 it is easy to check the following:

Lemma 2. *For $g \in G(\mathbf{A})$ and $h \in H(\mathbf{A})$ we have*

$$\omega(g_1)L(h) = L(h)\omega(\begin{pmatrix} 1 & 0 \\ 0 & \lambda(g)^{-1} \end{pmatrix} g).$$

Let $K \subset H(\mathbf{A}_f)$ be a compact open subgroup and write $K_1 = K \cap H_1(\mathbf{A}_f)$. Consider a K_1-invariant Schwartz function $\varphi \in S(V(\mathbf{A}_f)^n)$. Let

$$\tilde{\varphi}_\infty \in [S(V(\mathbf{R})^n) \otimes \Omega^{d-i}(D)]^{H_1(\mathbf{R})} \tag{11}$$

be a holomorphic Schwartz form and write $\tilde{\varphi} = \varphi \otimes \tilde{\varphi}_\infty$.

Note that for any $h_f \in H(\mathbf{A}_f)$ we have an embedding $j(h_f) : S^1_{h_f K h_f^{-1} \cap H_1(\mathbf{A}_f)} \hookrightarrow S_K$ with a corresponding map on cohomology groups (see e.g. [Har87] Section 5.1).

Definition 3. Let $[\eta] \in H^i_c(S_K, \mathbf{C})$ with central character χ_η trivial on $Z(H(\mathbf{R}))$. For $g \in G(\mathbf{A})^+$ let $h \in H(\mathbf{A})$ be any element such that $\lambda(h) = \lambda(g)$. (Note that $\lambda(g_\infty) > 0$ since $s \neq t$.) Put

$$K_1^{h_f} = h_f K h_f^{-1} \cap H_1(\mathbf{A}_f) = h_f K_1 h_f^{-1}$$

and

$$\tilde{\varphi}' = L(h_f \lambda(h_\infty^{1/2} I_m))\tilde{\varphi} \in [S(V(\mathbf{A})^n) \otimes \Omega^{d-i}(D)]^{H_1(\mathbf{R}) \times K_1^{h_f}}.$$

Now define a function on $G(\mathbf{A})^+$ by

$$\theta_\varphi(\eta)(g) = \langle [j(h_f)^*\eta], [\tilde{\varphi}'] \rangle_{K_1^{h_f}}(g_1). \tag{12}$$

Proposition 4. *The definition (12) is independent of the choice of h, left-invariant under $G(\mathbf{Q})^+$, and extends uniquely to a $G(\mathbf{Q})$-left invariant function on $G(\mathbf{A})$ with support in $G(\mathbf{Q})G(\mathbf{A})^+$. This extended function, also denoted by $\theta_\varphi(\eta)$, is an automorphic form on $G(\mathbf{A})$ with central character $\chi_V^n \chi_\eta$. Furthermore, $\theta_\varphi(\eta)$ defines a holomorphic Siegel modular form on $G(\mathbf{A})$ of weight $m/2$.*

Proof. It is easy to check that $\tilde{\varphi}'$ is still a holomorphic Schwartz form and gives rise to a class in

$$H^{d-i}_{\mathrm{ct}}(H_1(\mathbf{R}), S(V(\mathbf{A})^n)^K)^{\mathfrak{q}}_{\chi_m},$$

so one can apply the results of Kudla and Millson about the holomorphicity of the theta lift (see [KM90] Theorem 1(i) and Lemma 3.3).

As noted above, in terms of relative Lie algebra cocycles, the integral (12) is given by

$$\int_{H_1\mathbf{Q}_+ \backslash H_1(\mathbf{A})/K_1^{h_f} K_\infty} \eta(h_1 h_f) \wedge \theta(g_1, L(h_1 h_f \lambda(h_\infty)^{1/2} I_m)\tilde{\varphi}) dh_1(\mathbf{1}_{\mathfrak{p}}).$$

The right-invariance of the measure shows that the definition is independent of the choice of h. Furthermore, writing $h_\infty = h^1_\infty \lambda(h_\infty)^{1/2} I_m$ we can make a change of variables on D by $L_{h_\infty^{-1}} : z \mapsto z' = (h^1_\infty)^{-1} z$ and use the invariance of η under $Z(H(\mathbf{R})) = \mathbf{R}^*$ to rewrite the integral as

$$\int_{H_1\mathbf{Q}_+\backslash H_1(\mathbf{A})/hK_1K_\infty h^{-1}} DL^*_{h_\infty^{-1}}(\eta(h_1 h) \wedge \theta(g_1, L(h_1 h)\tilde{\varphi}))dh_1(DL_{h_\infty}\mathbf{1}_{\mathfrak{p}}).$$

The statement about the central character is now proven as in Lemma 5.1.9(ii) of [HK92]. To prove that $\theta_\varphi(\eta)$ is left-invariant under $\gamma \in \mathrm{GSp}_n(\mathbf{Q})^+$ let $\gamma' \in H(\mathbf{Q})_+$ such that $\lambda(\gamma') = \lambda(\gamma)$. One can now follow the proof of [HK92] Lemma 5.1.9(i), using the left-invariance of η under $H(\mathbf{Q})_+$ and the fact that for $\tilde{\varphi} \in [S(V(\mathbf{A}^n))^K \otimes \Lambda^{d-i}\mathfrak{p}^*]^{K_\infty}$ we have

$$\sum_{x \in V(\mathbf{Q})^n} L(\gamma')\tilde{\varphi}(x) = \sum_{x \in V(\mathbf{Q})^n} \tilde{\varphi}(x).$$

□

4 Fourier expansion

4.1 Definition of weighted cycles

We recall the following definitions from [Kud97]. Let $U \subset V$ be a $\mathbf{Q}$-subspace with $\dim_{\mathbf{Q}} U = n$ such that $(\, , \,)|_U$ is positive definite. Put $D_U = \{z \in D | z \perp U\}$ and let $H_{1,U}$ be the stabilizer of U in $H_1(\mathbf{R})$. Let $H^0_{1,U}$ be the connected component of identity of $H_{1,U}$. The orientation on D induces one on D_U as follows (see [KM90] pp. 130/1): We have a canonical isomorphism $T_z(D_U) \cong \mathrm{Hom}(z, z^\perp \cap U^\perp)$. Then $T_z(D_U)$ receives an orientation by the rule that the orientation of $T_z(D_U)$ followed by the orientation of $\mathrm{Hom}(z, U)$ is the orientation of $T_z(D) \cong \mathrm{Hom}(z, z^\perp)$.

For a fixed neat level $K \subset H_1(\mathbf{A}_f)$ and an element $h \in H_1(\mathbf{A}_f)$, we define a connected cycle associated to U as follows. Set

$$\Gamma'_h = H_1(\mathbf{Q})_+ \cap hKh^{-1}$$

and

$$\Gamma'_{h,U} = \Gamma'_h \cap H^0_{1,U}.$$

Let Γ_h (resp. $\Gamma_{h,U}$) denote the image of Γ'_h (resp. $\Gamma'_{h,U}$) in $H_1^{\mathrm{ad}}(\mathbf{R})^+$ (resp. $H^{\mathrm{ad}}_{1,U}(\mathbf{R})^+$).

Define the map

$$\Gamma_{h,U}\backslash D_U \to \Gamma_h\backslash D : \Gamma_{h,U} z \mapsto \Gamma_h z.$$

We will denote by $c(U, h, K)$ the connected cycle this defines in $H_{(s-n)t}(\Gamma_h\backslash D, \mathbf{Z})$. Note, that if $\gamma \in H_1(\mathbf{Q})_+$, then

$$\gamma \cdot c(U, h, K) = c(\gamma U, \gamma h, K),$$

where $\gamma \cdot c(U, h, K)$ denotes the cycle which is the image of the composite mapping

$$\Gamma_{h,U}\backslash D_U \to \Gamma_h\backslash D \to \gamma\Gamma_h\gamma^{-1}\backslash D.$$

Also note that

$$c(U, hk, K) = c(U, h, K)$$

for all $k \in K$.

We will now define weighted sums of these connected cycles to define cycles for $S^1_K = \coprod \Gamma_j\backslash D$. Let $\beta = \beta^t \in M_n(\mathbf{Q})$ be a positive definite symmetric matrix and, for $x = (x_1, \ldots, x_n) \in V^n$, represent the associated Gram matrix by

$$(x, x) = ((x_i, x_j)) \in \mathrm{Sym}_n(\mathbf{Q}),$$

where $(\, , \,)$ is the symmetric bilinear form on V. Let

$$\Omega_\beta = \{x \in V^n | \frac{1}{2}(x, x) = \beta\}$$

be the corresponding hyperboloid.

Let $S(V(\mathbf{A}_f^n))_{\mathbf{Z}}$ be the space of locally constant $\mathbf{Z}$-valued functions on $V(\mathbf{A}_f^n)$ of compact support. For any commutative ring R, let

$$S(V(\mathbf{A}_f)^n)_R = S(V(\mathbf{A}_f^n))_{\mathbf{Z}} \otimes_{\mathbf{Z}} R.$$

Motivated by [Kud97] Proposition 5.4 we make the following definition:

Definition 5. For $K \subset H_1(\mathbf{A}_f)$ and $\varphi \in S(V(\mathbf{A}_f)^n)_R$, a K-invariant Schwartz function, let

$$Z(\beta, \varphi, K) = \sum_j \sum_{x \in \Omega_\beta(\mathbf{Q}) \mod \Gamma'_j} \varphi(h_j^{-1}x) \cdot c(U(x), h_j, K) \in H_{(s-n)t}(S_K^1, \partial S_K^1, R),$$

where $U(x)$ is the $\mathbf{Q}$-subspace of V spanned by the components of x.

4.2 Fourier coefficients

The choice of maximal compact subgroup $K_\infty \subset H_1(\mathbf{R})$ and corresponding base point $z_0 \in D$ determines a positive definite form $(\ ,\ )_0$ on V which is a minimal majorant of the given form $(\ ,\ )$ on V of signature (s,t). We define the Gaussian $\varphi_0 \in S(V(\mathbf{R})^n)$ by

$$\varphi_0(x_1, \dots, x_n) = \prod_{i=1}^n \exp(-\pi(x_i, x_i)_0).$$

Kudla and Millson define in [KM90] a particular holomorphic Schwartz class $[\varphi_{nt}^+] \in H_{\mathrm{ct}}^{nt}(H_1(\mathbf{R}), S(V(\mathbf{R})^n))_{\chi_m}^{\mathfrak{q}}$ taking value in $\mathbf{S}(V(\mathbf{R})^n)$. Here the *polynomial Fock space* $\mathbf{S}(V(\mathbf{R})^n)$ is defined to be the space of those Schwartz functions on $V(\mathbf{R})^n$ of the form $p(v_1, \dots, v_n)\varphi_0(v_1, \dots, v_n)$, where $p(v_1, \dots, v_n)$ is a polynomial function on $V(\mathbf{R})^n$.

Following [KM82] and [FM02] (4.17) we give here the definition of the Schwartz form φ_n^+ in the case of signature $(s,1)$ (we refer the reader to §5 of [KM90] for the general case): For $x = (x_1, \dots, x_n) \in V(\mathbf{R})^n$ the Schwartz form $\varphi_n^+ \in [S(V(\mathbf{R})^n) \otimes \Omega^n(D)]^{H_1(\mathbf{R})} \cong [S(V(\mathbf{R})^n) \otimes \bigwedge^n(\mathfrak{p})^*]^{K_\infty}$ is given by

$$\varphi_n^+(x, w) = 2^{n/2} \det(x, w)\varphi_0(x)$$

for $w = (w_1, \dots, w_n) \in \mathfrak{p}^n \cong (z_0^\perp)^n$, where (x,w) is the matrix with (i,j)-th entry (x_i, w_j). (In fact, this differs from the definition in [KM90] by the factor $2^{n/2}$.) Theorem 5.2 of [KM90] proves that this gives rise to a holomorphic Schwartz class in $H_{\mathrm{ct}}^n(H_1(\mathbf{R}), S(V(\mathbf{R})^n))_{\chi_m}^{\mathfrak{q}}$.

Using the calculation of Fourier coefficients by Kudla and Millson we are going to prove in the next sections that for

$$\tilde{\varphi}_\infty = \varphi_{nt}^+ \in [S(V(\mathbf{R})^n) \otimes \Omega^{nt}(D)]^{H_1(\mathbf{R})} \cong [S(V(\mathbf{R})^n) \otimes \bigwedge^{nt}(\mathfrak{p})^*]^{K_\infty}$$

the form $\theta_\varphi(\eta)$ given by Proposition 4 is an arithmetic (e.g. algebraic, rational or p-integral) Siegel modular form for arithmetic φ and η. We fix this choice for $\tilde{\varphi}_\infty$ from now on.

Remark. We note that the action of $Z(H(\mathbf{R})) = \mathbf{R}^*$ on $\tilde{\varphi}_\infty$ used in Definition 3 does not preserve the property of taking values in $\mathbf{S}(V(\mathbf{R})^n)$. This requires us to restrict to $G_1(\mathbf{R}) \times G(\mathbf{A}_f)^+$ in Theorem 6, but this is sufficient for our analysis of the arithmetic of the Fourier-Jacobi expansion in Section 5.1.

Let $(V_+, (\ ,\ )_+)$ be a positive definite inner product space of dimension m over $\mathbf{R}$, and let $\varphi_+^0 \in S(V_+^n)$ be the Gaussian

$$\varphi_+^0(x) = \exp(-\pi \mathrm{tr}(x, x)_+).$$

If $x \in V_+^n$ with $(1/2)(x,x)_+ = \beta \in \mathrm{Sym}_n(\mathbf{R})$, then for $g \in G_1(\mathbf{R})$, we define the generalized Whittaker function

$$W_\beta(g) = \omega_+(g)\varphi_+^0(x), \tag{13}$$

where ω_+ is the Weil representation associated to V_+. As in [Kud97] (7.22) we note that for

$$g = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & \\ & v^{-1/2} \end{pmatrix} \ell$$

with $\ell \in L_\infty$ and $\tau = u + iv = g(i \cdot 1_n) \in \mathbf{H}_n$ we have

$$W_\beta(g) = \det(v)^{m/4} \exp(2\pi i \mathrm{tr}(\beta\tau)) \det(\ell)^{m/2}.$$

Theorem 6. *Let $K \subset H(\mathbf{A}_f)$ be a compact open subgroup and put $K_1 = K \cap H_1(\mathbf{A}_f)$. Let $[\eta] \in H_c^{(s-n)t}(S_K, \mathbf{C})$ and $\varphi \in S(V(\mathbf{A}_f)^n)^{K_1}$. For $g \in G_1(\mathbf{R}) \times G(\mathbf{A}_f)^+$ let $h \in H(\mathbf{A}_f)$ such that $\lambda(h) = \lambda(g)$ and*

$$\varphi' = \omega(g_f, h)\varphi \in S(V(\mathbf{A}_f)^n).$$

Then we have

$$\theta_\varphi(\eta)(g) = c_K * \sum_{\beta > 0} W_\beta(g_\infty) \cdot \int_{Z(\beta, \varphi', K_1^h)} \eta(h),$$

where $c_K = \begin{cases} 2, & \text{if } -1 \in K \cap H_1(\mathbf{Q})_+; \\ 1, & \text{else.} \end{cases}$

Proof. We follow the proof of Theorem 8.1 of [Kud97]. Put $\eta' = j(h)^* \eta$. Write $H_1(\mathbf{A}_f) = \coprod_j H_1(\mathbf{Q})_+ h_j K_1^h$. By (3) we get

$$\theta_\varphi(\eta) = \int_{S^1_{K_1^h}} \eta' \wedge \theta(g_\infty, \varphi')$$

$$= \sum_j \int_{\Gamma_j \backslash D} \eta'(h_j) \wedge \theta(g_\infty, h_j, \varphi') = \sum_j \sum_{\beta \in \mathrm{Sym}_n(\mathbf{Q})} \sum_{x \in \Omega_\beta(\mathbf{Q}) \mod \Gamma'_j} \int_{\Gamma_{j,x} \backslash D} \eta'(h_j) \wedge \omega(g_\infty) \tilde{\varphi}'(h_j^{-1} x).$$

Here we use the fact, that if $\Gamma_{j,x}$ is the image in Γ_j of the stabilizer $\Gamma'_{j,x}$ of x in Γ'_j, then if $-1 \notin K \cap H_1(\mathbf{Q})_+$ we have

$$\Gamma'_{j,x} \backslash \Gamma'_j \cong \Gamma_{j,x} \backslash \Gamma_j.$$

Now one main result of [KM90] is that the terms where β is not positive definite vanish (this is where our assumption that t is odd comes in!). We require a form of Thom's Lemma, as stated in [KM90] Theorem 9.1, where the results of [KM87] (for $\Gamma \backslash D$ compact) and [KM88] (for $\Gamma \backslash D$ finite volume) are combined. In fact, these results do not cover the case of an infinite geodesic, which can arise for signature $(s, 1)$. [FM02] added a proof for this in the case of signature $(2, 1)$ and a principal congruence subgroup. We are going to prove this for our main case of interest ($(s, t) = (3, 1)$ and $n = 2$) in the next section.

Theorem 7 (Thom's Lemma)**.** *Let $\beta > 0$ and $x \in \Omega_\beta(\mathbf{Q})$. Put $U = U(x)$. Let Γ_U be a discrete subgroup of $H^0_{1,U}$. For any closed and bounded $(s-n)t$-form η on $\Gamma_U \backslash D$,*

$$\int_{\Gamma_U \backslash D} \eta \wedge (\omega(g_\infty) \tilde{\varphi}_\infty)(x) = W_\beta(g_\infty) \int_{\Gamma_U \backslash D_U} \eta$$

□

Remark. Kudla and Millson also treat the case of even t. In this case one has non-trivial Fourier coefficients for positive semi-definite β, for which the period integral involves also powers of an Euler form (see Theorem 9.3 of [KM90]).

4.3 Thom Lemma for hyperbolic 3-space

In this section we will prove Thom's Lemma in the special case we are most interested in: fix $(s,t)=(3,1)$, $n=2$ throughout this section.

Let $F=Q(\sqrt{-D})$ be an imaginary quadratic field. We denote its ring of integers by $\mathcal{O}$. Underlying our calculation is the accidental isomorphism

$$\mathrm{Spin}_V(\mathbf{R}) \cong \mathrm{Res}_{F/\mathbf{Q}}\mathrm{SL}_2(\mathbf{R}).$$

On the one hand, the symmetric space D in this case can be realized as

$$D=\{Z\in V(\mathbf{R}) : (Z,Z)=-1\}^0,$$

on the other hand it is isomorphic to hyperbolic 3-space $\mathbf{H}_3=\mathbf{C}\times\mathbf{R}_{>0}$, elements of which we write as (z,r) with $z=x+iy$ for $x,y\in\mathbf{R}, r\in\mathbf{R}_{>0}$. The group $\mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{H}_3$ via hyperbolic isometries. The action is most concisely written using quaternion notation: identifying the point (z,r) with the quaternion $q=z+rj$ the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : q\mapsto \frac{aq+b}{cq+d}.$$

Since we only work with $V(\mathbf{R})$ in the following, we can assume without loss of generality that V is the hermitian matrices

$$V=\{X\in M_2(F) : X^t=\overline{X}\}$$

with quadratic form

$$X\mapsto -\det(X)$$

and corresponding bilinear form

$$(X,Y)\mapsto -\frac{1}{2}\mathrm{tr}(X\cdot Y^*),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

The group $\mathrm{SL}_2(F)$ acts isometrically on V by

$$X\mapsto gX\overline{g}^t$$

for $X\in V$ and $g\in \mathrm{SL}_2(F)$.

We fix an orthonormal basis of $V(\mathbf{R})$ given by $e_1=\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}$, $e_2=\begin{pmatrix} & 1 \\ 1 & \end{pmatrix}$, $e_3=\begin{pmatrix} & i \\ -i & \end{pmatrix}$, and $e_4=\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}=Z_0$, with respect to which the pairing is of the form

$$(\ ,\ )\sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -1 \end{pmatrix}.$$

We identify $\mathfrak{p}$ with $\mathbf{R}^3$ via the basis $\{e_1,e_2,e_3\}$ for $Z_0^\perp$. Let $\{\omega_1,\omega_2,\omega_3\}$ be the corresponding dual basis of $\mathfrak{p}^*$. Choosing the basis $\{e_1,\dots,e_4\}$ fixes an isomorphism $V(\mathbf{R})^2\cong M_{4,2}(\mathbf{R})$. By [FM02] (4.14) we then obtain for $X=((x_{1,1},\dots x_{1,4}),(x_{2,1},\dots x_{2,4}))\in V(\mathbf{R})^2$ that

$$\varphi_2^+(X)=2(\omega(1,X)\wedge\omega(2,X))\cdot\varphi_0(X)$$

with $\omega(s,X)=\sum_{i=1}^3 x_{i,s}\omega_i$.

We also pick two isotropic vectors $u_0 = \begin{pmatrix} 1 & \\ & \end{pmatrix}$ and $u_0' = \begin{pmatrix} & \\ & 1 \end{pmatrix}$ and note that with respect to the basis $\{u_0, e_2, e_3, u_0'\}$ of $V(\mathbf{R})$ the majorant associated to the base point Z_0 is given by

$$(\, , \,)_0 \sim \begin{pmatrix} 1/2 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1/2 \end{pmatrix}.$$

For an ideal $\mathfrak{n} \subset \mathcal{O}$ put

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) : c \in \mathfrak{n} \right\}.$$

Let $[\eta] \in H^1_c(\Gamma_0(\mathfrak{n})\backslash \mathbf{H}_3, \mathbf{C})$ be the class of a rapidly decreasing harmonic form η. Writing $\varphi(X,(z,r))$ for the 2-form on D corresponding to φ_2^+ we need to show

$$\int_{\Gamma_U \backslash D} \eta(z,r) \wedge \varphi(X,(z,r)) = \exp(-\pi \mathrm{tr}(X,X)) \int_{\Gamma_U \backslash D_U} \eta$$

for $U = U(X)$ with $(X,X) > 0$ and Γ_U the stabilizer of U in $\Gamma_0(\mathfrak{n})$. The space $X^\perp$ has signature $(1,1)$ and so the stabilizer Γ_U is either infinitely cyclic or trivial. In the first case, the cycle $\Gamma_U \backslash D_U$ is a closed geodesic and Thom's Lemma holds by [KM88]. When the stabilizer is trivial, the cycle D_U is an infinite geodesic joining two cusps.

Theorem 8. *Assume Γ_U is trivial. Then*

$$\int_D \eta \wedge \varphi(X) = \exp(-\pi \mathrm{tr}(X,X)) \int_{D_U} \eta. \tag{14}$$

Proof. We essentially carry out a two-dimensional version of the proof for signature $(2,1)$ in Theorem 7.6 of [FM02] (which additionally treats the case of an Eisenstein cohomology class η). The final integral that needs to be solved is complicated by the fact that the Fourier expansion for the modular forms over imaginary quadratic fields involve K-Bessel functions.

We can assume

$$X = (2au_0 + be_2, 2cu_0 + de_3)$$

with $d = d'/\sqrt{D}$ and $a, c \in \mathbf{Q}$, $b, d' \in \mathbf{Q}_{>0}$ so that D_U is the geodesic joining the cusps ∞ and $\frac{a}{b} + \frac{c}{d'}\sqrt{-D} \in F$. The stabilizer of the cusp ∞ is $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix} : \alpha \in \mathcal{O} \right\}$. We claim that

$$\int_D \eta \wedge \varphi(X) = \int_{\Gamma_\infty \backslash D} \eta \wedge \sum_{\alpha \in \mathcal{O}} \varphi(X + (2b\alpha_1 u_0, 2d\alpha_2 u_0), (z,r)),$$

where we write $\alpha_1 = \mathrm{Re}(\alpha)$ and $\alpha_2 = \mathrm{Im}(\alpha)$. We will, in fact, consider the holomorphic function

$$I(s) := \int_{\Gamma_\infty \backslash D} \eta \wedge \sum_{\gamma \in \Gamma_\infty} \gamma^*(y^s \varphi(X,(z,r)))$$

for $s \in \mathbf{C}$, do the unfolding for $\mathrm{Re}(s) \gg 0$, and evaluate the integral in the end at $s = 1$ by analytic continuation.

We get an explicit formula for $\varphi(\begin{pmatrix} 1 & \alpha \\ & 1 \end{pmatrix}.X, (z,r))$ by calculating

$$\begin{aligned}
&\varphi\left(\begin{pmatrix} r^{-1/2} & -zr^{-1/2} \\ & r^{1/2} \end{pmatrix}.(X + (2b\alpha_1 u_0, 2d\alpha_2 u_0)), (0,1) \right) = \\
&= \frac{1}{2} e^{-\frac{2\pi}{r^2}(a+b(\alpha_1 - x))^2} e^{-\pi(b^2+d^2)} e^{-\frac{2\pi}{r^2}(c+d(\alpha_2-y))^2} \cdot \left(\frac{1}{r}(a+b(\alpha_1-x))\frac{dr}{r} + b\frac{dx}{r}\right) \wedge \left(\frac{1}{r}(c+d(\alpha_2-y))\frac{dr}{r} + d\frac{dy}{r}\right) \\
&=: (\varphi_1 dy \wedge dr + \varphi_2 dx \wedge dy + \varphi_3 dr \wedge dx)\, e^{-\pi(b^2+d^2)}.
\end{aligned}$$

The Fourier transform with respect to α, which we define as

$$\hat{\varphi}(\beta_1+i\beta_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(\begin{pmatrix}1 & \alpha_1+i\alpha_2\\ & 1\end{pmatrix}.X,(z,r))e^{2\pi i\alpha_1\beta_1}e^{2\pi i\alpha_2\beta_2}d\alpha_1 d\alpha_2$$

is then given by

$$\hat{\varphi_1}(\beta,X)dy\wedge dr+\hat{\varphi_2}(\beta,X)dx\wedge dy+\hat{\varphi_3}(\beta,X)dr\wedge dx$$

with

$$\begin{aligned}\hat{\varphi_2}(\beta,X) &= e^{-\frac{\pi}{2}\left(\frac{\beta_1 r}{b}\right)^2}e^{-\frac{\pi}{2}\left(\frac{\beta_2 r}{d}\right)^2}e^{2\pi i\beta_1 x}e^{2\pi i\beta_2 y}e^{-2\pi i\beta_1\frac{a}{b}}e^{-2\pi i\beta_2\frac{c}{d}},\\ \hat{\varphi_1}(\beta,X) &= -\frac{i\beta_1 r}{2b^2}\cdot\hat{\varphi_2}(\beta,X),\\ \hat{\varphi_3}(\beta,X) &= -\frac{i\beta_2 r}{2d^2}\cdot\hat{\varphi_2}(\beta,X).\end{aligned}$$

Remark. The definition of the cycles $Z(\beta,\varphi,K)$ in Definition 5 and the proof of Theorem 6 reduced us to working with just one connected component, so in the following we will assume for simplicity that the class number of F is one. For appropriate definitions of the automorphic forms in the general case, in particular, the expression of their L-value in terms of period integrals of the form $\sum_j\int_{c(U(X),h_j,K)}\eta$, we refer the reader to [Kur78] Theorem 2.

By the Eichler-Shimura-Harder isomorphism $[\eta]$ can be represented by a $\Gamma_0(\mathfrak{n})$-invariant harmonic differential on $\mathbf{H}_3$ of the form $-f_0\frac{dz}{r}+f_1\frac{dr}{r}+f_2\frac{d\overline{z}}{r}$, where

$$f=(f_0,f_1,f_2):\mathbf{H}_3\to\mathbf{C}^3$$

is a weight 2 cusp form for $\Gamma_0(\mathfrak{n})$ (see [Urb95] Théorème 3.2, [Har87] §3.6 and [CW94] (2.1)). By [CW94] (with a correction from [Byg99] Proposition 100) such a function f has a Fourier expansion (about the cusp $(0,\infty)$) of the form

$$f(z,r)=\sum_{\xi\in\vartheta^{-1}}c(\xi)r^2\mathbf{K}(4\pi|\xi|r)\cdot\mathrm{diag}(\xi/|\xi|,1,\overline{\xi}/|\xi|)\Psi(\xi z),$$

where ϑ is the different of F, $\mathbf{K}(t)$ is the vector-valued K-Bessel function

$$\mathbf{K}(t)=(-\frac{1}{2}iK_1(t),K_0(t),\frac{1}{2}iK_1(t)),$$

and $\Psi(z)=\exp(-2\pi i(z+\overline{z}))$.

By Poisson summation we therefore obtain

$$I(s)=\int_D\eta\wedge\varphi(X)=\int_{\Gamma_\infty\backslash D}\mathrm{vol}(\mathbf{C}/\mathcal{O})^{-1}\sum_{\beta\in\vartheta^{-1}}((-f_0+f_2)\hat{\varphi_1}(\beta,X)+f_1\hat{\varphi_2}(\beta,X)-i(f_0+f_2)\hat{\varphi_3}(\beta,X))r^{2s-2}dx\wedge dy\wedge dr.$$

We pick a fundamental domain for $\Gamma_\infty\backslash D$ and integrate with respect to $z=x+iy$, which singles out the Fourier coefficients of f:

$$I(s)=e^{-\pi(b^2+d^2)}\sum_{\xi\in\vartheta^{-1}}c(\xi)\Psi\left(\xi(\frac{a}{b}+i\frac{c}{d})\right)\int_0^\infty r^{2s-1}e^{-2\pi((\frac{\xi_1 r}{b})^2+(\frac{\xi_2 r}{d})^2)}\left(K_0(4\pi|\xi|r)+\left(\frac{\xi_1^2}{b^2}+\frac{\xi_2^2}{d^2}\right)\frac{r}{|\xi|}K_1(4\pi|\xi|r)\right)dr.$$

Changing the variable of integration to $t=|\xi|r$ this equals

$$e^{-\pi(b^2+d^2)}L(f,\Psi_X,s)\int_0^\infty t^{2s-1}e^{-2\pi((\frac{\xi_1 r}{|\xi|b})^2+(\frac{\xi_2 r}{|\xi|d})^2)}\left(K_0(4\pi t)+\left(\frac{\xi_1^2}{b^2}+\frac{\xi_2^2}{d^2}\right)\frac{t}{|\xi|^2}K_1(4\pi t)\right)dr,$$

where for $\mathrm{Re}(s)>3/2$

$$Z(f,\Psi_X,s)=\sum_{\xi\in\vartheta^{-1}}c(\xi)\Psi\left(\xi(\frac{a}{b}+i\frac{c}{d})\right)\mathrm{Nm}(\xi)^{-s}$$

is a zeta-function of f twisted by the additive character Ψ_X, which has an analytic continuation to $s \in \mathbf{C}$ (see [CW94] Proposition 2.1). If f is a so-called "plusform" (see [CW94] Section 2.4), i.e. is invariant under

$$\Gamma_0^+(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}) : c \in \mathfrak{n} \right\}$$

then this zeta function can be related to the the usual L-series of f:

$$Z(f, 1, s) = |\mathcal{O}^*||\mathrm{disc}(F)| \times \sum_{\mathfrak{m}} c(\mathfrak{m})\mathrm{Nm}(\mathfrak{m})^{-s},$$

where $\mathfrak{m}$ runs over the nonzero ideals of $\mathcal{O}$.

We now specialize to $s = 1$ and use integration by parts for

$$\int_0^\infty re^{-2\pi((\frac{\xi_1 r}{b})^2 + (\frac{\xi_2 r}{d})^2)} \left(\frac{\xi_1^2}{b^2} + \frac{\xi_2^2}{d^2} \right) \frac{r}{|\xi|} K_1(4\pi|\xi|r)dr$$

and the following two properties of the K-Bessel function (see [MOS66] 3.1.1 and 3.2):

$$\frac{d}{dx}(xK_1(x)) = -xK_0(x)$$

and

$$\lim_{x \to 0^+} xK_1(x) = 1.$$

We conclude that

$$I(1) = \frac{1}{(4\pi)^2} \exp(-\pi \mathrm{tr}(X, X)) \cdot Z(f, \Psi_X, 1),$$

which (by a slight generalisation of) [CW94] Proposition 2.1 agrees with the right-hand side of (14) . □

Remark 9. *Using Gauss sums and the orthogonality of Dirichlet characters, these special values of additive twists can be expressed as finite sums of special values of twists by Dirichlet characters (see e.g. Theorem 1 of [Koj97]). This implies that the corresponding Fourier coefficients in Theorem 6 are given by sums of special L-values of the cuspform (for* GL_2 *over the imaginary quadratic field) twisted by Dirichlet characters. This is analogous to the statement in Theorem 2 of [Koj97] about the Fourier coefficients of the Shintani lifting from classical elliptic modular forms to half-integral weight modular forms. We plan to investigate the Fourier coefficients of the theta lift considered in this section further. In particular, this opens up the possibility of deciding the non-vanishing of the Kudla-Millson theta lift in this situation (c.f. the proof of [HST93] Proposition 3).*

5 Arithmetic properties

5.1 Arithmetic Siegel modular forms

For $\tau = u + iv \in \mathbf{H}_n$ and $g_f \in G(\mathbf{A}_f)$ we define

$$\theta_\varphi(\eta)(\tau, g_f) = \det(v)^{-m/4} \cdot \theta_\varphi(\eta)(g_\tau g_f),$$

where $g_\tau = \begin{pmatrix} v^{1/2} & uv^{-1/2} \\ 0 & v^{-1/2} \end{pmatrix} \in G_1(\mathbf{R})$.

We recall from [Har84] (2.2.2.1) that any such holomorphic automorphic form ϕ on $\mathbf{H}_n \times (G(\mathbf{A}_f)/L)$ has a Fourier-Jacobi expansion of the form

$$\phi(\tau, g_f) = \sum_{\beta \in \mathrm{Sym}_n(\mathbf{Q})} a_\beta(\tau, g_f),$$

where

$$a_\beta(\tau, g_f) = \int_{U(\mathbf{Q})\backslash U(\mathbf{A})} \phi(u(\tau, g_f))\exp(-2\pi i \mathrm{tr}(\beta u))du.$$

One checks (see e.g. [Sug85](1-19)) that

$$a_\beta(\tau, g_f) = a_\beta(g_f) \cdot \exp(2\pi i \mathrm{tr}(\beta\tau))$$

for some $a_\beta(g_f) \in \mathbf{C}$.

Let p be any rational prime. Fix embeddings $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$. Let $E \subset \overline{\mathbf{Q}}$ be a number field and $\wp$ the prime of E above p. By the q-expansion principle (see [FC90]) arithmetic holomorphic automorphic forms are characterized by their Fourier-Jacobi expansion. We therefore call f *algebraic* (resp. E-rational, resp. $\wp$-integral) if $a_\beta(g)$ lies in $\overline{\mathbf{Q}}$ (resp. E, resp. $\mathcal{O}_{E_\wp}$) for all $\beta \in \mathrm{Sym}_n(\mathbf{Q})$ and all $g \in G(\mathbf{A}_f)$. (In fact, one only needs to consider finitely many $g \in G(\mathbf{A}_f)$, see [Har84] §3 or [Tay91] §3.)

Since $R(\mathbf{A}_f)$ preserves $S(V(\mathbf{A}_f)^n)_{\overline{\mathbf{Q}}}$, Proposition 4 and Theorem 6 imply:

Corollary 10. *If* $[\eta] \in H_c^{(s-n)t}(S_K, \overline{\mathbf{Q}})$ *and* $\varphi \in S(V(\mathbf{A}_f)^n)_{\overline{\mathbf{Q}}}$ *a* K_1*-invariant Schwartz function then* $\theta_\varphi(\eta)(\tau, g_f)$ *is an algebraic holomorphic Siegel modular form of weight* $m/2$*.*

Remark. We can, in fact, replace $\overline{\mathbf{Q}}$ by $\mathbf{Q}^{\mathrm{ab}}$ in the above statement since the Weil representation is defined over $\mathbf{Q}^{\mathrm{ab}}$. In the next section we will give an example of a suitable choice of φ taking values in $\mathcal{O}_{E_\wp}$ for which we can prove the $\wp$-integrality of $\theta_\varphi(\eta)$ for $\eta \in \mathrm{im}(H_c^{(s-n)t}(S_K, \mathcal{O}_{E_\wp}) \to H_c^{(s-n)t}(S_K, E_\wp))$.

5.2 Definition of Schwartz function

Definition 11. For any prime ℓ and any lattice $L \subset V \otimes \mathbf{Q}_\ell$ define its *norm* to be the $\mathbf{Z}_\ell$-module generated by $\{(x,x) | x \in L\}$ and its *(Hessian) dual lattice* by $L^\sharp = \{x \in V \otimes \mathbf{Q}_\ell | 2(x,y) \in \mathbf{Z}_\ell \, \forall y \in L\}$. We call a lattice $L \subset V \otimes \mathbf{Q}_\ell$ *maximal* if its norm equals $\mathbf{Z}_\ell$ and there exists no other lattice containing L of this norm.

Let X be an integral lattice on V and put $X_\ell = X \otimes_{\mathbf{Z}} \mathbf{Z}_\ell$ for every prime ℓ of $\mathbf{Z}$. Let (ℓ^{-n_ℓ}) be the norm of $X_\ell^\sharp$ and set

$$N = N(X) = \prod_\ell \ell^{n_\ell}$$

(the "*level of the lattice* X"). Furthermore, let ℓ^{m_ℓ} be the smallest power such that $\ell^{m_\ell} X_\ell^\sharp \subset X_\ell$ and put

$$M = M(X) = \prod_{\ell \neq 2} \ell^{m_\ell} \times \max\{8, 2^{m_2}\}.$$

Note that all prime divisors of N also divide M.

To avoid trivial non-vanishing of our theta lift due to the symmetry of the archimedean Schwartz function φ_n^+ (e.g. for n even and $t = 1$ we have $\varphi_n^+((x_1, \ldots, x_n)) = -\varphi((-x_1, \ldots, x_n)))$ we do not take the finite Schwartz function just as the product of the characteristic functions of the lattices X_ℓ^n. Following standard practice in the classical case, we instead introduce the following adelic congruence condition:

Definition 12. Let $v = (v_\ell) \in V(\mathbf{A}_f)^n$ with

$$\begin{cases} v_\ell = 0 & \text{for } \ell \nmid M \\ v_\ell \in (X_\ell^\sharp)^n - X_\ell^n, & \text{for } \ell \mid M. \end{cases}$$

Define

$$\varphi = \varphi_v = \prod_\ell \varphi_{v,\ell} \in S(V(\mathbf{A}_f)^n)_{\mathbf{Z}}$$

by putting φ_ℓ equal to the characteristic function of $v_\ell + X_\ell^n$ for all ℓ.

We fix $K = \prod_\ell K_\ell \subset H(\mathbf{A}_f)$ with

$$K_\ell = \{h \in H(\mathbf{Q}_\ell) | hX_\ell = X_\ell, h \equiv 1 \mod \ell^{m_\ell} M_m(\mathbf{Z}_\ell)\} \text{ for all } \ell$$

and note that φ is invariant under K.

this could of course be improved for particular choices of v, e.g. just involving one prime in v

COMMENT

Remark 13. *An alternative solution would be to introduce a congruence condition at an auxiliary prime $q \nmid 2N$, following [KM90] p.133: Choose $v \in X_q^n$ such that $v_i \not\equiv -v_i \mod qX_q$ for $i = 1 \dots n$ and set φ_q equal to the characteristic function of $v + qX_q^n$. The drawback of this approach is that it introduces q^2 into the level of the theta lift.*

We define

$$L^0(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\hat{\mathbf{Z}}) | C \equiv 0 \mod N(X)M_n(\hat{\mathbf{Z}}) \right\} \subset G(\mathbf{A}_f).$$

and

$$L^1(N,M) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\hat{\mathbf{Z}}) | C \equiv 0_n \mod N(X)M_n(\hat{\mathbf{Z}}), A \equiv I_n \mod M(X)M_n(\hat{\mathbf{Z}}) \right\} \subset G(\mathbf{A}_f)$$

and put

$$L^1(N) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G(\hat{\mathbf{Z}}) | C \equiv 0_n \mod N(X)M_n(\hat{\mathbf{Z}}), A \equiv D \equiv I_n \mod N(X)M_n(\hat{\mathbf{Z}}) \right\} \subset L^1(N,N).$$

It is easy to see that $\lambda(K) \subset \hat{\mathbf{Z}}^*$. More precisely, we have the following two lemmas:

Lemma 14 ([Eic74] Satz 11.2, [Yos84] Lemma 1.4)**.** *Assume that X_ℓ is a maximal lattice. If $h \in H(\mathbf{Q}_\ell)$ satisfies $\lambda(h) \in \mathbf{Z}_\ell^*$ then there exists $k \in \{h \in H(\mathbf{Q}_\ell) | hX_\ell = X_\ell\}$ such that $\lambda(k) = \lambda(h)$.*

Lemma 15. *For $\ell \mid M$ we have*

$$\lambda(K_\ell) \supseteq \{z \in \mathbf{Z}_\ell^* : z \equiv 1 \mod M\mathbf{Z}_\ell\}.$$

Proof. For any $\lambda \in \mathbf{Z}_\ell^*, \lambda \equiv 1 \mod M\mathbf{Z}_\ell$, Hensel's lemma tells us that we can find a square root of λ in $\mathbf{Z}_\ell$ congruent to $1 \mod M\mathbf{Z}_\ell$ and so an element in $K_\ell \cap Z(H(\mathbf{Q}_q))$ with similitude λ. □

Recall from Section 2.1 the quadratic character χ_V associated to V and take p and E as in Section 5.1.

Theorem 16. *Let $[\eta] \in \mathrm{im}(H_c^{(s-n)t}(S_K, \mathcal{O}_{E_\wp}) \to H_c^{(s-n)t}(S_K, E_\wp))$. Assume that $\lambda(H(\mathbf{Q}_p)) \supset \mathbf{Z}_p^*$. Then $\theta_\varphi(\eta)$ is a $\wp$-integral holomorphic Siegel modular form of weight $m/2$ and central character χ_V^n, and level N and character χ_V in the sense that*

$$\theta_\varphi(\eta)(gk) = \chi_V(\det A)\theta_\varphi(\eta)(g) \text{ for } k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L^1(N,M) \cap G(\mathbf{A}_f)^+, \lambda(k) \equiv 1 \mod M\hat{\mathbf{Z}}. \qquad (15)$$

Proof. We first note that for $\ell \nmid M$, X_ℓ is a maximal lattice by [Kit93] Lemma 5.2.1 since $X_\ell = X_\ell^\sharp$ implies $dX_\ell \in \mathbf{Z}_\ell^*$ (notation as in Kitagawa). By the definition of $\theta_\varphi(\eta)$ it suffices to check (15) for $g \in G(\mathbf{A})^+$. Note that under our assumption on the lattice X and the preceding lemmata we then have $\theta_\varphi(\eta)(gk) = \theta_\varphi(\eta)(gk_1)$ by Definition 3. Therefore the statement about the level and character follows from the following Lemma:

Lemma 17. *For $v \in V(\mathbf{A}_f)^n$ as above, we get*

$$\omega(k_1)\varphi_v = \chi_V(\det A)\varphi_{vA^{-1}} \text{ for } k_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L^0(N) \cap G_1(\mathbf{A}_f).$$

Proof. [Yos84] Lemma 2.1 proves this statement for all places $\ell \nmid M$.

For $\ell \mid N$ we can adopt his argument as follows: Since X_ℓ is integral, we have

$$\omega(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix})\varphi_\ell = \varphi_\ell \text{ for } u \in M_n(\mathbf{Z}_\ell), {}^t u = u$$

by (5). By (6) we see that

$$\omega(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix})\varphi_{v,\ell} = \chi_{V,\ell}(\det a)\varphi_{va^{-1},\ell} \text{ for } a \in \mathrm{GL}_n(\mathbf{Z}_\ell).$$

Since

$$\hat{\varphi}_{v,\ell}(x) = \int_{v_\ell + X_\ell^n} \psi_\ell(\mathrm{tr}(x,y))dy = \psi_\ell(\mathrm{tr}(x,v_\ell)) \int_{X_\ell^n} \psi_\ell(\mathrm{tr}(x,y))dy = \mathrm{vol}(X_\ell^n)\psi_\ell(\mathrm{tr}(x,v_\ell))\mathbf{1}_{(X_\ell^\sharp)^n}$$

we deduce from (5) that

$$\omega(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\varphi_\ell = \omega(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix})\varphi_\ell \text{ for } u \in \ell M_n(\mathbf{Z}_\ell), {}^t u = u$$

which implies

$$\omega(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix})\varphi_\ell = \varphi_\ell \text{ for } u \in \ell M_n(\mathbf{Z}_\ell), {}^t u = u.$$

Now the assertion follows from the Iwahori decomposition of $L^0(N)$. □

By strong approximation we have $G(\mathbf{A}) = G(\mathbf{Q})L^1(N,M)G(\mathbf{R})_+$ and so it suffices to check $\wp$-integrality on $L^1(N,M)$, and by definition of θ on $L^1(N,M) \cap G(\mathbf{Q})G(\mathbf{A})^+$.

If $k \in L^1(N,M) \cap G(\mathbf{A}_f)^+$ and $\lambda(k) \equiv 1 \mod M\hat{\mathbf{Z}}$ this follows directly from (15) (which reduces us to considering $\theta_\varphi(\eta)(1)$) and Theorem 6. In general, if $k \in L^1(N,M) \cap G(\mathbf{Q})G(\mathbf{A})^+$ write

$$k = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} g z_\infty^{-1} z_\infty$$

for $g \in G(\mathbf{A})^+$, $\lambda \in \mathbf{Q}_{>0}$ with $|\lambda|_p = 1$ and $z_\infty = \lambda^{-1/2} I_m \in G(\mathbf{R})^+$. This is possible since $\lambda(k) \in \hat{\mathbf{Z}}^*$ and our assumption that $\lambda(H(\mathbf{Q}_p)) \supset \mathbf{Z}_p^*$. Since the central character is trivial on $Z(H(\mathbf{R}))$ we then have $\theta_\varphi(\eta)(k) = \theta_\varphi(\eta)(g z_\infty^{-1})$ and by Theorem 6 this equals

$$\sum_{\beta > 0} W_\beta(g_\infty z_\infty^{-1}) \cdot \int_{Z(\beta,\varphi',K_1^h)} \eta(h)$$

for $h \in H(\mathbf{A}_f)$ with $\lambda(h) = \lambda(g z_\infty^{-1})$ and $\varphi' = \omega(g_f, h)\varphi = L(h)\omega(\begin{pmatrix} 1 & 0 \\ 0 & \lambda(k)^{-1} \end{pmatrix} k)\varphi$, the latter equality using Lemma 2. Applying Lemma 17 for $\begin{pmatrix} 1 & 0 \\ 0 & \lambda(k)^{-1} \end{pmatrix} k$ and noting $|\lambda(h_p)|_p = 1$ we deduce the $\wp$-integrality of $\theta_\varphi(\eta)$. □

Remark. Assume that for $\ell \nmid Nq$ the form η is an eigenfunction for the Hecke algebra $\mathcal{H}(H_1(\mathbf{Q}_\ell)//H_1(\mathbf{Z}_\ell))$ and that the Witt index of $V \otimes \mathbf{Q}_\ell$ (the dimension of the maximal isotropic subspace) is less than or equal to n. Then Rallis' generalisation of the Eichler commutation relation ([Ral82] §4.B) implies that $\theta_\varphi(\eta)|_{G_1(\mathbf{A})}$ is an eigenfunction for the Hecke algebra $\mathcal{H}(G_1(\mathbf{Q}_\ell)//G_1(\mathbf{Z}_\ell))$.

5.3 Orthogonal Spaces of dimension 4

In this section we restrict to quadratic spaces V with dimension $m = 4$ and signature $(3,1)$ and analyze the Hecke properties of the theta lift.

We refer the reader to Section 2 of [Rob01] for a summary of results of four dimensional quadratic spaces. We are interested in the following examples: Let F be an imaginary quadratic field with ring of integers $\mathcal{O}$ and discriminant d_F and for every place v of F write $\mathfrak{q}_v \subset \mathcal{O}$ for the corresponding prime ideal. For B_0 a quaternion algebra over $\mathbf{Q}$ put $B = B_0 \otimes_\mathbf{Q} F$, write $*$ for the main involution of B and denote the natural extension of the non-trivial automorphism of F over $\mathbf{Q}$ to the semi-automorphism of B by $-$. Put

$$V = \{x \in B | \overline{x} = x^*\}$$

with quadratic form $N(x) = xx^*$. This is a four dimensional quadratic space of signature $(3,1)$ since B necessarily splits at ∞.

Let

$$N_1 = \prod_{B_0 \text{ramified at } \ell} \ell.$$

Note that B is ramified exactly at all the places above factors of N_1 which split in $F/\mathbf{Q}$. If ℓ splits in $F/\mathbf{Q}$ then $V \otimes \mathbf{Q}_\ell$ is isomorphic to $B_{0,\ell}$. If B_0 is split at a prime ℓ which is inert or ramified in $F/\mathbf{Q}$ (with $F \otimes \mathbf{Q}_\ell = \mathbf{Q}_\ell(\sqrt{d})$, say) then $V \otimes \mathbf{Q}_\ell$ is isomorphic to (see e.g. [Rob01] p.273)

$$\left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \overline{e} \end{pmatrix} \middle| f, g \in \mathbf{Q}_\ell, e \in F \otimes \mathbf{Q}_\ell \right\} \subset M_2(F \otimes \mathbf{Q}_\ell).$$

If B_0 is ramified at a prime ℓ which is inert or ramified in $F/\mathbf{Q}$ then $V \otimes \mathbf{Q}_\ell$ is isomorphic to (see e.g. [Rob01] p.273)

$$\left\{ \begin{pmatrix} f & -\delta e \\ \overline{e} & g \end{pmatrix} \middle| f, g \in \mathbf{Q}_\ell, e \in F \otimes \mathbf{Q}_\ell \right\} \subset M_2(F \otimes \mathbf{Q}_\ell),$$

where δ is a representative for the nontrivial coset of $\mathbf{Q}_\ell^*/\mathrm{Nm}_{\mathbf{Q}_\ell}^{F \otimes \mathbf{Q}_\ell}((F \otimes \mathbf{Q}_\ell)^*)$.

Let $N_2 \in \mathbf{Z}$ be squarefree and coprime to N_1. We consider an Eichler order $R \subset B$ of level $N_1 N_2 \mathcal{O}$, where R_v is a maximal order in B_v for all $v \nmid N_2$ and R_v is conjugate to

$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_v) \middle| c \equiv 0 \mod \mathfrak{q}_v \mathcal{O}_v \right\}$$

for $v \mid N_2$ (where B_v has been identified with $M_2(F_v)$). Following Yoshida [Yos80] we define the lattice

$$X := \{x \in R | \overline{x} = x^*\} \subset V.$$

For simplicity we make the following assumption from now on:

Assumption 18. *Assume that d_F is coprime to $N_1 N_2$, i.e. B_0 is split at primes that ramify in $F/\mathbf{Q}$ and N_2 is not divisible by ramified primes.*

For ease of reference we record the following lemma:

Lemma 19. *Under assumption 18 the level of the lattice X equals $N(X) = N_1 N_2 d_F$ and $M(X) = 8 N_1 N_2 d_F$ if d_F odd and $M(X) = 2 N_1 N_2 d_F$ if d_F even.*

Proof. We need to consider the following cases for the lattices X_ℓ (as before, we denote the norm of $X_\ell^\sharp$ by $\ell^{n_\ell}\mathbf{Z}_\ell$ and let ℓ^{m_ℓ} be the smallest power such that $\ell^{m_\ell} X_\ell^\sharp \subset X_\ell$):

(a) ℓ split, B_0 split: In this case X_ℓ is isomorphic to either $M_2(\mathbf{Z}_\ell)$ or the Eichler order in $M_2(\mathbf{Z}_\ell)$and $n_\ell = m_\ell = 0$ and 1, respectively.

(b) $\ell \mid N_1$ split, B_0 ramified: In this case X_ℓ is isomorphic to the maximal order $R_{0,\ell}$ in the division quaternion algebra. The dual lattice with respect to the reduced trace is given by $X_\ell^\sharp = \pi_\ell^{-1} R_{0,\ell}$ for a prime element π_ℓ in $R_{0,\ell}$, $n_\ell = 1$ (see e.g. [Vig80] Corollaire II.1.7) and $m_\ell = 1$.

(c) $\ell \nmid N_2$ inert, B_0 split: In this case we have the self-dual (with respect to $2(X,Y) = \mathrm{tr}(XY^*)$) maximal order $X_\ell = \left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \overline{e} \end{pmatrix} | f, g \in \mathbf{Z}_\ell, e \in \mathcal{O}_\ell \right\}$, so $n_\ell = m_\ell = 0$.

(d) $\ell \mid N_1$ inert, B_0 ramified: Here $X_\ell = \left\{ \begin{pmatrix} f & -\delta e \\ \overline{e} & g \end{pmatrix} \middle| f, g \in \mathbf{Z}_\ell, e \in \mathcal{O}_\ell \right\}$ with dual

$$X_\ell^\sharp = \left\{ \begin{pmatrix} f & -\delta e \\ \overline{e} & g \end{pmatrix} \middle| f, g \in \mathbf{Z}_\ell, e \in \ell^{-1}\mathcal{O}_\ell \right\},$$

so $n_\ell = m_\ell = 1$.

(e) $\ell \mid N_2$ inert, B_0 split: In this case we have the order $X_\ell = \left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \overline{e} \end{pmatrix} \middle| f \in \mathbf{Z}_\ell, g \in \ell\mathbf{Z}_\ell, e \in \mathcal{O}_\ell \right\}$ with dual $X_\ell^\sharp = \left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \overline{e} \end{pmatrix} \middle| f \in \ell^{-1}\mathbf{Z}_\ell, g \in \mathbf{Z}_\ell, e \in \mathcal{O}_\ell \right\}$, so $n_\ell = m_\ell = 1$.

(f) $\ell \nmid N_2$ ramified in $F/\mathbf{Q}$, B_0 split. Again $X_\ell = \left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \overline{e} \end{pmatrix} \middle| f, g \in \mathbf{Z}_\ell, e \in \mathcal{O}_\ell \right\}$, but now $X_\ell^\sharp = \left\{ \begin{pmatrix} e & f\sqrt{d} \\ g\sqrt{d} & \overline{e} \end{pmatrix} \middle| f, g \in \ell^{-1}\mathbf{Z}_\ell, e \in \ell^{-1/2}\mathcal{O} \otimes \mathbf{Z}_\ell \right\}$ so $n_\ell = m_\ell = 1$.

□

As mentioned before we are mainly interested in the case $n = 2$. In this case we claim that $L^1(N(X)) \cap G(\mathbf{A}_f)^+$ differs from $L^1(N(X))$ only at the places ℓ ramified in $F/\mathbf{Q}$: By [Rob96] Section 3 $G(\mathbf{Q}_\ell) \neq G(\mathbf{Q}_\ell)^+$ for a 4-dimensional quadratic space W over $\mathbf{Q}_\ell$ only if its Witt index is 1, i.e. if $W \cong k \perp \mathbf{H}$ for $k/\mathbf{Q}_\ell$ a quadratic extension and $\mathbf{H}$ the hyperbolic plane over $\mathbf{Q}_\ell$. Since for split ℓ the space $V \otimes \mathbf{Q}_\ell$ in our case is either $M_2(\mathbf{Q}_\ell)$ or the division quaternion algebra over $\mathbf{Q}_\ell$ this means that $\lambda(H(\mathbf{Q}_\ell)) = \mathbf{Q}_\ell^*$ and $G(\mathbf{Q}_\ell) = G(\mathbf{Q}_\ell)^+$. From the explicit description of $V \otimes \mathbf{Q}_\ell$ given above for inert or ramified ℓ one deduces the Witt decomposition $V \otimes \mathbf{Q}_\ell \cong (F \otimes \mathbf{Q}_\ell) \perp \mathbf{H}$. This implies (see [Rob96] Section 3) that $\lambda(H(\mathbf{Q}_\ell)) = \mathrm{Nm}_{\mathbf{Q}_\ell}^{\mathbf{Q}_\ell \otimes F}((\mathbf{Q}_\ell \otimes F)^*)$, so for ℓ inert in $F/\mathbf{Q}$ we have $G(\mathbf{Z}_\ell) \subset G(\mathbf{Q}_\ell)^+$, whereas $[G(\mathbf{Z}_\ell) : G(\mathbf{Q}_\ell)^+ \cap G(\mathbf{Z}_\ell)] = 2$ for ℓ ramified in $F/\mathbf{Q}$.

Together with Lemma 19 this claim implies the following Corollary to Theorem 16 (note that for $k \in L^1(N, N)$, $\lambda(k) \equiv 1 \mod N$ is equivalent to $D \equiv 1 \mod N$ since $A^t D - B^t C = \lambda(\begin{pmatrix} A & B \\ C & D \end{pmatrix}) I_n$):

Corollary 20. *Assume $p \nmid \mathrm{disc}(F/\mathbf{Q})$ and $n = 2$. Choose $v \in V(\mathbf{A}_f)^n$ as in Definition 12 and define φ and K accordingly. Let $[\eta] \in \mathrm{im}(H_c^{(s-n)t}(S_K, \mathcal{O}_{M_\wp}) \to H_c^{(s-n)t}(S_K, M_\wp))$. Then $\theta_\varphi(\eta)$ is a $\wp$-integral holomorphic Siegel modular form of weight 2 and trivial central character, and level $N = N_1 N_2 d_F$ and character χ_V in the sense that*

$$\theta_\varphi(\eta)(gk) = \chi_V(\mathrm{det} A)\theta_\varphi(\eta)(g) \text{ for } k = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L^1(N). \tag{16}$$

Next we will analyse the Hecke action on our theta lift. This can easily be done for general n, so we allow again $n = 1, 2$ or 3. First note that B^* defines an algebraic group over F. Denote its restriction of scalars from F to $\mathbf{Q}$ by $\tilde{B}^*$. It acts on V via $x \mapsto gx\overline{g}^*$. We refer the reader to [HST93] §1 for a description of the relationship between cuspidal automorphic forms (denoted by $\mathbf{A}$ in the following) of $\mathrm{GO}(V)_\mathbf{A} = H_\mathbf{A}$ and $\tilde{B}_\mathbf{A}^*$ for $B = M_2(F)$. One can generalize this to all our quaternion algebras B using the characterisation of the special similitude group $\mathrm{GSO}(V)$ given in [Rob01] Theorem 2.3 and Proposition 2.7. We are only going to use that an automorphic form for $H_\mathbf{A}$ represented by $[\eta]$ arises from an automorphic form $f_\eta \in \mathcal{A}(\tilde{B}_\mathbf{A}^*)$ (not uniquely, compare [HST93] Proposition 2).

For every place v of F such that $v \nmid N$ we define Hecke operators $T'(v)$ acting on $f_\eta \in \mathcal{A}(\tilde{B}_\mathbf{A}^*)$ as follows: We may assume that $R_v = R \otimes_\mathcal{O} \mathcal{O}_v$ is mapped onto $M_2(\mathcal{O}_v)$ when we fix a splitting $B \otimes F_v \cong M_2(F_v)$. Let π_v be a prime element of F_v and let $R_v^* \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} R_v^* = \bigcup_i h_i R_v^*$. Then define

$$(T'(v)f_\eta)(h) = \sum_i f_\eta(hh_i) \text{ for } h \in \tilde{B}_\mathbf{A}^*.$$

For split ℓ we identify $V \otimes \mathbf{Q}_\ell \cong B_{0,\ell}$ and note that $\mathrm{GSO}(B_{0,\ell}) \cong (B_{0,\ell} \times B_{0,\ell})/\mathbf{Q}_\ell$, with (h_1, h_2) acting by $x \mapsto h_1 x h_2^{-1}$ (see [BSP91] p. 60). For π a prime element of $B_{0,\ell}$ we denote by $(\pi, 1)$ the element of $H(\mathbf{Q}_\ell)$ that acts on $x \in V \otimes \mathbf{Q}_\ell$ by $x \mapsto \pi x$.

Now consider the Hecke action on Siegel modular forms. Let $L^1(N) = \prod_\ell L^1(N)_\ell$ for $L^1(N)_\ell \subset G(\mathbf{Z}_\ell)$ with $L^1(N)_\ell = G(\mathbf{Z}_\ell)$ for $\ell \nmid N$. For any ℓ and $M \in G(\mathbf{Q}_\ell)$ the double coset $L^1(N)_\ell M L^1(N)_\ell = \bigcup_i g_i L^1(N)_\ell$ acts on $\phi : G(\mathbf{A}) \to \mathbf{C}$ by $(L^1(N)_\ell M L^1(N)_\ell)\phi(g) = \sum_i \phi(gg_i)$. We single out the operators

$$T_\ell = L^1(N)_\ell \mathrm{diag}(\underbrace{\ell, \ldots, \ell}_n, \underbrace{1, \ldots, 1}_n) L^1(N)_\ell$$

and

$$R_\ell^{(s)} = L^1(N)_\ell \mathrm{diag}(\underbrace{\ell,\dots,\ell}_{n-s},\underbrace{1,\dots,1}_{s},\underbrace{\ell,\dots,\ell}_{n-s},\underbrace{\ell^2,\dots,\ell^2}_{s}) L^1(N)_\ell \text{ for } 0 \le s \le n.$$

Theorem 21. *Let $[\eta] \in H_c^{(s-n)t}(S_K, \mathbf{C})$ correspond to $f_\eta \in \mathcal{A}(\tilde{B}^*_{\mathbf{A}})$. Consider $\ell \nmid N_2$ unramified in $F/\mathbf{Q}$. If $\ell \nmid N_1$ then assume that f_η is an eigenform for the Hecke operator $T'(v)$ with eigenvalue λ_v for all $v \mid \ell$. If $\ell \mid N_1$ then assume that $[\eta]$ is an eigenform with eigenvalue ± 1 for the Atkin-Lehner involution given by right multiplication by $(\pi, 1) \in H(\mathbf{Q}_\ell)$ for π a prime element of $B_{0,\ell}$.*

Then $\theta_\varphi(\eta)$ is an eigenfunction for the Hecke operators T_ℓ and $R_\ell = R_\ell^{(1)}$ with eigenvalues for $n = 2$ given by:

	T_ℓ	R_ℓ
$(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ *split*, $\ell \nmid N$	$\ell(\lambda_{\mathfrak{l}} + \lambda_{\bar{\mathfrak{l}}})$	$(\ell^2 - 1) + \ell\lambda_{\mathfrak{l}}\lambda_{\bar{\mathfrak{l}}}$
ℓ *inert*, $\ell \nmid N$	0	$(\ell^2 + 1) + \ell\lambda_\ell$
ℓ *split*, $\ell \mid N_1$	$\pm\ell$	$\ell(\ell + 1)$

Proof. For $\ell \nmid N$ this can be deduced from [HST93] Lemmata 10,11, where we set the auxiliary $\delta_\ell = +1$ to ensure local non-vanishing. For similar calculations see [Yos80] Theorem 5.2, [BSP91] Theorem 6.1, and [Urb98] Théorème 3.3.5.

For $\ell \mid N_1$ split in $F/\mathbf{Q}$ we have $B_\ell \cong B_{0,\ell} \times B_{0,\ell}$ and X_ℓ is therefore isomorphic to the maximal order $R_{0,\ell}$ in the division quaternion algebra $B_{0,\ell}$. We can therefore refer to the (local) calculation of the R_ℓ eigenvalue in [BSP91] Lemma 7.3 (b), after multiplying by $\mathrm{diag}(\ell^{-1},\dots,\ell^{-1}) \in Z(G(\mathbf{Q}))$ and observing that $\chi_{V,\ell} = 1$.

We calculate the T_ℓ action by first checking that

$$L^1(N)_\ell \mathrm{diag}(\underbrace{\ell,\dots,\ell}_{n},\underbrace{1,\dots,1}_{n}) L^1(N)_\ell = \bigcup_B \begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix} L^1(N)_\ell,$$

where B runs over the $\ell^{n(n+1)/2}$ representatives for the symmetric matrices modulo ℓ. To prove this, check that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} \ell I_n & 0 \\ 0 & I_n \end{pmatrix} = \begin{pmatrix} \ell I_n & X \\ 0 & I_n \end{pmatrix}\begin{pmatrix} A - CX & \frac{1}{\ell}(B - XD) \\ \ell C & D \end{pmatrix}$$

and note that note that for $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in L^1(N)_\ell$ we can find X symmetric modulo ℓ such that $B \equiv XD$ mod ℓ (use that D is invertible modulo ℓ, $A^tD - B^tC = \lambda(\begin{pmatrix} A & B \\ C & D \end{pmatrix})I_n$ and $A^tB = B^tA$).

By definition of $\theta_\varphi(\eta)$ it suffices to consider the Hecke action when evaluating on $g \in G(\mathbf{A})^+$. Let $h \in H(\mathbf{A})$ with $\lambda(h) = \lambda(g)$ and $h' \in G(\mathbf{Q}_\ell)$ with $\lambda(h') = \ell$. We get

$$\theta_\varphi(\eta)(g\begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix}) = \int_{S^1_{Khh'}} \eta(hh') \wedge \theta(\left(g\begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix}\right)_1, L(hh')\tilde{\varphi}).$$

By [HK92] Lemma 5.1.7 (a) we have

$$\theta(\left(g\begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix}\right)_1, L(hh')\tilde{\varphi}) = \theta(g_1, L(h)\omega(\begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix}_1)L(h')\tilde{\varphi}).$$

By Lemma 2 we know that $\omega(\begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix}_1)L(h') = L(h')\omega(\begin{pmatrix} \ell I_n & B \\ 0 & \ell^{-1} I_n \end{pmatrix})$. As in the proof of [BSP91] Lemma 7.3 (a) we calculate

$$\omega(\begin{pmatrix} \ell I_n & B \\ 0 & \ell^{-1} I_n \end{pmatrix})\varphi(x) = \frac{1}{\ell^{2n}}\psi_\ell(\mathrm{tr}(\ell B(x,x)))\varphi(x\begin{pmatrix} \ell & 0 \\ 0 & \ell \end{pmatrix}) = \frac{1}{\ell^{2n}}\varphi(\pi x).$$

We now assume that $h' = (\pi, 1)$, by which we denoted the element of $H(\mathbf{Q}_\ell)$ that acts on $x \in V \otimes \mathbf{Q}_\ell$ by $x \mapsto \pi x$, and so $L(h')\varphi(\pi x) = \ell^2\varphi(x)$. Since $\pi R_{0,\ell}\pi^{-1} = R_{0,\ell}$ we have $K_f^{hh'} = K_f^h$ and hence

$$\theta_\varphi(\eta)(g \begin{pmatrix} \ell I_n & B \\ 0 & I_n \end{pmatrix}) = \frac{1}{\ell^{2(n-1)}}\theta_\varphi(\eta((\pi, 1)))(g).$$

□

References

[AK10] Mahesh Agarwal and Kris Klosin, *Yoshida lifts and the Bloch-Kato conjecture for the convolution L-function*, Preprint.

[BDSP10] Siegfried Böcherer, Neil Dummigan, and Rainer Schulze-Pillot, *Yoshida lifts and Selmer groups*, J. Math. Soc. Japan (2010), to appear.

[BSP91] Siegfried Böcherer and Rainer Schulze-Pillot, *Siegel modular forms and theta series attached to quaternion algebras*, Nagoya Math. J. **121** (1991), 35–96.

[Byg99] Jeremy Bygott, *Modular Forms and Modular Symbols over imaginary quadratic fields*, Ph.D. thesis, University of Exeter, 1999, available for download at `http://www.warwick.ac.uk/staff/J.E.Cremona/`.

[CW94] John E. Cremona and Elise Whitley, *Periods of cusp forms and elliptic curves over imaginary quadratic fields*, Math. Comp. **62** (1994), no. 205, 407–429.

[Eic74] Martin Eichler, *Quadratische Formen und orthogonale Gruppen*, Springer-Verlag, Berlin, 1974, Zweite Auflage, Die Grundlehren der mathematischen Wissenschaften, Band 63.

[FC90] Gerd Faltings and Ching-Li Chai, *Degeneration of abelian varieties*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 22, Springer-Verlag, Berlin, 1990, With an appendix by David Mumford.

[FM02] Jens Funke and John Millson, *Cycles in hyperbolic manifolds of non-compact type and Fourier coefficients of Siegel modular forms*, Manuscripta Math. **107** (2002), no. 4, 409–444.

[FM06] ______, *Cycles with local coefficients for orthogonal groups and vector-valued Siegel modular forms*, Amer. J. Math. **128** (2006), no. 4, 899–948.

[Har84] Michael Harris, *Eisenstein series on Shimura varieties*, Ann. of Math. (2) **119** (1984), no. 1, 59–94.

[Har87] Günther Harder, *Eisenstein cohomology of arithmetic groups. The case* GL_2, Invent. Math. **89** (1987), no. 1, 37–118.

[HK92] Michael Harris and Stephen S. Kudla, *Arithmetic automorphic forms for the nonholomorphic discrete series of* $\mathrm{GSp}(2)$, Duke Math. J. **66** (1992), no. 1, 59–121.

[HST93] Michael Harris, David Soudry, and Richard Taylor, *l-adic representations associated to modular forms over imaginary quadratic fields. I. Lifting to* $\mathrm{GSp}_4(\mathbf{Q})$, Invent. Math. **112** (1993), no. 2, 377–411.

[Kit93] Yoshiyuki Kitaoka, *Arithmetic of quadratic forms*, Cambridge Tracts in Mathematics, vol. 106, Cambridge University Press, Cambridge, 1993.

[KM82] Stephen S. Kudla and John J. Millson, *Geodesic cyclics and the Weil representation. I. Quotients of hyperbolic space and Siegel modular forms*, Compositio Math. **45** (1982), no. 2, 207–271. MR MR651982 (83m:10037)

[KM86] ———, *The theta correspondence and harmonic forms. I*, Math. Ann. **274** (1986), no. 3, 353–378.

[KM87] ———, *The theta correspondence and harmonic forms. II*, Math. Ann. **277** (1987), no. 2, 267–314.

[KM88] ———, *Tubes, cohomology with growth conditions and an application to the theta correspondence*, Canad. J. Math. **40** (1988), no. 1, 1–37.

[KM90] ———, *Intersection numbers of cycles on locally symmetric spaces and Fourier coefficients of holomorphic modular forms in several complex variables*, Inst. Hautes Études Sci. Publ. Math. (1990), no. 71, 121–172.

[Koj97] Hisashi Kojima, *Fourier coefficients of modular forms of half integral weight, periods of modular forms and the special values of zeta functions*, Hiroshima Math. J. **27** (1997), no. 2, 361–371.

[Kud97] Stephen S. Kudla, *Algebraic cycles on Shimura varieties of orthogonal type*, Duke Math. J. **86** (1997), no. 1, 39–78.

[Kur78] P. F. Kurčanov, *The cohomology of discrete groups and Dirichlet series that are related to Jacquet-Langlands cusp forms*, Izv. Akad. Nauk SSSR Ser. Mat. **42** (1978), no. 3, 588–601.

[MOS66] Wilhelm Magnus, Fritz Oberhettinger, and Raj Pal Soni, *Formulas and theorems for the special functions of mathematical physics*, Third enlarged edition. Die Grundlehren der mathematischen Wissenschaften, Band 52, Springer-Verlag New York, Inc., New York, 1966.

[Pra06] Kartik Prasanna, *Integrality of a ratio of Petersson norms and level-lowering congruences*, Ann. of Math. (2) **163** (2006), no. 3, 901–967.

[Pra08] ———, *Arithmetic aspects of the theta correspondence and periods of modular forms*, Eisenstein series and applications, Progr. Math., vol. 258, Birkhäuser Boston, Boston, MA, 2008, pp. 251–269.

[Pra09] ———, *Arithmetic properties of the Shimura-Shintani-Waldspurger correspondence*, Invent. Math. **176** (2009), no. 3, 521–600, With an appendix by Brian Conrad.

[Ral82] Stephen Rallis, *Langlands' functoriality and the Weil representation*, Amer. J. Math. **104** (1982), no. 3, 469–515.

[Rob96] Brooks Roberts, *The theta correspondence for similitudes*, Israel J. Math. **94** (1996), 285–317.

[Rob01] ———, *Global L-packets for* GSp(2) *and theta lifts*, Doc. Math. **6** (2001), 247–314 (electronic).

[Sug85] Takashi Sugano, *On holomorphic cusp forms on quaternion unitary groups of degree* 2, J. Fac. Sci. Univ. Tokyo Sect. IA Math. **31** (1985), no. 3, 521–568.

[Tay91] Richard Taylor, *Galois representations associated to Siegel modular forms of low weight*, Duke Math. J. **63** (1991), no. 2, 281–332.

[Urb95] Eric Urban, *Formes automorphes cuspidales pour* GL_2 *sur un corps quadratique imaginaire. Valeurs spéciales de fonctions L et congruences*, Compositio Math. **99** (1995), no. 3, 283–324.

[Urb98] ———, *Module de congruences pour* GL(2) *d'un corps imaginaire quadratique et théorie d'Iwasawa d'un corps CM biquadratique*, Duke Math. J. **92** (1998), no. 1, 179–220.

[Vig80] Marie-France Vignéras, *Arithmétique des algèbres de quaternions*, Lecture Notes in Mathematics, vol. 800, Springer, Berlin, 1980.

[Yos80] Hiroyuki Yoshida, *Siegel's modular forms and the arithmetic of quadratic forms*, Invent. Math. **60** (1980), no. 3, 193–248.

[Yos84] ———, *On Siegel modular forms obtained from theta series*, J. Reine Angew. Math. **352** (1984), 184–219.

[Yos86] ———, *The action of Hecke operators on theta series*, Algebraic and topological theories (Kinosaki, 1984), Kinokuniya, Tokyo, 1986, pp. 197–238.