An Eisenstein ideal for imaginary quadratic fields

by

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ABSTRACT

For certain algebraic Hecke characters χ of an imaginary quadratic field F we define an Eisenstein ideal in a Hecke algebra acting on cuspidal automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ and prove a lower bound for its index in terms of the special L-value $L^{\mathrm{alg}}(0,\chi)$. From this we obtain a lower bound for the size of the Selmer group of a p-adic Galois character associated to χ. The method we use is to show that p-divisibility of $L^{\mathrm{alg}}(0,\chi)$ implies a congruence mod p between a multiple of an Eisenstein cohomology class associated to χ (in the sense of G. Harder) and a cuspidal cohomology class in the cohomology of a hyperbolic 3-orbifold. Implementing this requires bounding the denominator of the Eisenstein cohomology class, which we do by analytic methods, and using the geometry of the Borel-Serre compactification of these spaces to control torsion in the compactly supported cohomology of degree 2. We then use the work of R. Taylor *et al.* on associating Galois representations to cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ to construct elements in Selmer groups.

CHAPTER I

Introduction

Many interesting results or conjectures in number theory connect analytic and algebraic objects: The analytic class number formula, Kummer's criterion, the BSD-conjecture, and the main conjectures of Iwasawa theory all relate certain L-values to sizes of (pieces of) class groups or, more generally, Selmer groups. In this thesis we prove an analogue for imaginary quadratic fields of results over $\mathbf{Q}$ of the following form:

"If p^n divides the L-value $L(1-k,\chi)$ for a Dirichlet character χ, then p^n divides the order of a Selmer group related to χ."

Results of this form have been proven for $\mathbf{Q}$ in a number of different ways (cf. [Ri], [MW], [HP], [Th], [Ru]). We obtain our results for imaginary quadratic fields by following a strategy going back to Ribet's proof of the converse to Herbrand's theorem [Ri] that Wiles extended in [W90] to prove the Main Conjecture of Iwasawa theory for Hecke characters of totally real fields. The idea is to use the p-divisibility of the L-value to produce congruences between an Eisenstein series associated to χ (and involving $L(0,\chi)$) and cuspforms, whose associated Galois representations then allow deductions about certain Selmer groups.

The congruences used by Ribet and Wiles are found in the integral structure of the q-expansions of modular forms. Skinner developed in [S02a] an approach based mainly on analytic and representation-theoretic techniques, avoiding the input from algebraic geometry available for $\mathrm{GL}_{2/\mathbf{Q}}$ and working instead with the integral structure coming from singular cohomology and making use of Harder's Eisenstein cohomology. It was suggested there that this method might extend to other reductive groups, even those where the associated symmetric spaces are not hermitian. Earlier, Harder and Pink [HP] also proved such a result for $\mathrm{GL}_{2/\mathbf{Q}}$ using Eisenstein cohomology.

We work here with with $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_{2/F})$ for an imaginary quadratic field F and consider an unramified algebraic Hecke character $\chi : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of Weil type (A_0). We define an Eisenstein ideal related to χ in a p-adic Hecke algebra acting on (cohomological) cuspidal automorphic forms of G and prove a lower bound for its index. This lower bound is given in terms of the value $L^{\mathrm{alg}}(0,\chi)$. We follow Skinner (cf. [S02a]) in using cohomological congruences in the proof of this result (see Theorem 1.1 below). In Chapter III we construct an Eisenstein cohomology class $\mathrm{Eis}\,\omega_\chi$ annihilated by the Eisenstein ideal and having integral "constant term". We show that p-divisibility of the L-value implies a congruence mod p between $\mathrm{Eis}\,\omega_\chi$, multiplied by its denominator, and a cuspidal cohomology class in the cohomology of certain adelic symmetric spaces attached to G. This requires bounding the denominator of the Eisenstein cohomology class $\mathrm{Eis}\,\omega_\chi$, which we do by integrating along suitable modular symbols (see Chapter IV). In deducing the existence of a congruence we encounter a problem that does not arise for $\mathrm{GL}_{2/\mathbf{Q}}$: torsion in the compactly supported cohomology of degree 2. In Chapter V we make a careful analysis of the restriction map to the cohomology of the boundary of the Borel-Serre compactification of the symmetric space. This gives us a criterion for deciding which classes lie in its image. Here we make use of a result of Serre for $\mathrm{SL}_2(\mathcal{O}_F)$ in [Se70], which we reinterprete in our context and extend to all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$.

As indicated above, one application of our bound for the index of the Eisenstein ideal is a lower bound for the size of the Selmer group of an infinite order p-adic Galois character associated to χ. This is carried out in Chapter VII. After finding the cohomological congruences described above, we use the "Eichler-Shimura-Harder isomorphism" to relate the cuspidal cohomology to cuspidal automorphic representations. Using techniques developed by Wiles, Urban, and Skinner we apply the results of Taylor *et al.* on associating Galois representations to cuspidal automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$ to bound the size of certain Selmer groups from below by $L^{\mathrm{alg}}(0,\chi)$. We get around the restriction on the central character that Taylor's result requires of the cuspidal representations by factoring our χ appropriately.

To give a more precise account, let F be any imaginary quadratic field different from $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$. We exclude these two fields here because the Eisenstein cohomology is trivial in the situation we consider. Let $\mathfrak{p}$ be a prime in F such that the underlying rational prime is greater than 3, splits in F, and such that p does not divide the class number of F. Fix an embedding $\overline{F}_{\mathfrak{p}} \hookrightarrow \mathbf{C}$.

Let $\chi : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ be an unramified Hecke character of infinity type z^2 (i.e. $\chi_\infty(z) = z^2$). Choose two characters $\mu_1, \mu_2 : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of infinity type z and z^{-1}, respectively, such that $\chi = \mu_1/\mu_2$. (This freedom- gained by going up to GL_2- will come in useful later!) Let $\mathcal{O}_\chi$ denote the ring of integers in a sufficiently large finite extension F_χ of $F_{\mathfrak{p}}$.

Denote by S the (adelic) locally symmetric space associated to G and a certain open compact subgroup K_f of $G(\mathbf{A}_f)$ depending on μ_1 and μ_2, by $\overline{S}$ the Borel-Serre compactification of S, and by ∂_S the boundary of $\overline{S}$. For the definition of these objects see Sections 2.3 and 2.8. Let $\mathbf{T}_\chi$ be the $\mathcal{O}_\chi$-subalgebra generated by the Hecke operators acting on the cuspidal cohomology of $\overline{S}$ with coefficients in F_χ. We call now the ideal $\mathbf{I}_{\mu_1,\mu_2} \subseteq \mathbf{T}_\chi$ generated by

$$\left\{T_v - \mu_{1,v}^{-1}(\mathfrak{P}_v) - \mu_{2,v}^{-1}(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v) : v \notin R\right\}$$

the Eisenstein ideal associated to (μ_1, μ_2), where $\mathfrak{P}_v$ denotes the maximal ideal in $\mathcal{O}_{F_v}$ and R is the finite set of places where the μ_i are ramified.

Our main result can now be stated as:

Theorem 1.1. *There exists an $\mathcal{O}_\chi$-algebra surjection*

$$\mathbf{T}_\chi/\mathbf{I}_{\mu_1,\mu_2} \twoheadrightarrow \mathcal{O}_\chi/\left(L^{\mathrm{alg}}(0,\chi)\right).$$

As indicated above, the proof of Theorem 1.1 breaks down into three parts: (1) construction of a suitable Eisenstein cohomology class, (2) bounding its denominator, and (3) dealing with torsion in $H^2_c(\overline{S},\mathcal{O}_\chi)$. Using Harder's theory of Eisenstein cohomology, as developed in [Ha79], [Ha82], [HaGL2], we associate to χ (really to the pair (μ_1,μ_2)) an explicit cohomology class ω_χ in $H^1(\partial_S,\mathcal{O}_\chi)$ and use Eisenstein summation to get a class $\operatorname{Eis}\omega_\chi$ in $H^1(\overline{S},\mathbf{C})$, and even in $H^1(\overline{S},F_\chi)$. Differing from the situation for $\mathbf{Q}$, the restriction of $\operatorname{Eis}\omega_\chi$ to the boundary is not ω_χ but $\omega_\chi - \frac{L(0,\overline{\chi})}{L(0,\chi)}\tilde{\omega}_\chi$ for a dual class $\tilde{\omega}_\chi$. This restriction is integral, though, if we assume that χ is anticyclotomic, by which we mean that $\chi^c(x) := \chi(\overline{x})$ equals $\overline{\chi}(x)$ for all $x \in F^*\backslash\mathbf{A}_F^*$. This is automatic for unramified Hecke characters (see Lemma 3.16).

We define the denominator of a class $c \in H^1(S,F_\chi)$ to be the ideal

$$\delta(c) = \{a \in \mathcal{O}_\chi : ac \in \operatorname{im}(H^1(S,\mathcal{O}_\chi) \to H^1(S,F_\chi))\}.$$

In Chapter IV we integrate $\operatorname{Eis}\omega_\chi$ along suitable cycles to bound its denominator from below. In fact, up to this point our methods can deal with any F (including $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$), and almost any anticyclotomic character χ of infinity type z^2 (and certain other cases; see Theorem 4.17):

Theorem 1.2. *Let χ be an anticyclotomic Hecke character of an imaginary quadratic field of infinity type z^2 (satisfying some mild condition on its conductor). Then $\delta(\operatorname{Eis}\omega_\chi) \subset (L^{\mathrm{alg}}(0,\chi))$.*

Bounding the denominator of the Eisenstein cohomology class is an interesting result in its own right and prior to our result only the case of unramified Hecke characters for $F = \mathbf{Q}(i)$ had been analyzed (see [Ko]). (For an analysis of denominators of Eisenstein cohomology for unramified characters in degree 2 see [F]). The cycles we use are motivated by the classical modular symbol: we essentially integrate along the path

$$\begin{aligned} \sigma : \mathbf{R}_{>0} &\to S \\ t &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix}. \end{aligned}$$

(This is only a relative cycle in $H_1(\overline{S}, \partial\overline{S}, \mathbf{Z})$; see Section 4.4 for how we use the integrals to bound the denominator of the Eisenstein cohomology class.) A rather involved adelic calculation shows that the result for this toroidal integral is

$$\int_\sigma \operatorname{Eis}\omega_\chi \sim \frac{L(0,\mu_1)L(0,\mu_2^{-1})}{L(0,\chi)},$$

the '$\sim$' indicating equality up to units in $\mathcal{O}_\chi$.

To extract $L^{\mathrm{alg}}(0,\chi)$ as the bound we use results by Hida and Finis on the non-vanishing modulo p of the L-values $L^{\mathrm{alg}}(0,\theta\mu_i^{\pm 1})$ as θ varies in an anticyclotomic $\mathbf{Z}_\ell$-extension for $\ell \neq p$. (Finis and Hida impose different conditions on the $\mu_i^{\pm 1}$, allowing for different cases of χ, one of them being anticyclotomic Hecke characters of infinity type z^2.) We replace $\operatorname{Eis}\omega_\chi$ by a "twisted" version $\operatorname{Eis}^\theta\omega_\chi$ for a finite order character θ such that $a\cdot\operatorname{Eis}^\theta\omega_\chi$ is integral if $a\cdot\operatorname{Eis}\omega_\chi$ is. Up to units the result is then

$$\int_\sigma \operatorname{Eis}^\theta\omega_\chi \sim \frac{L(0,\mu_1\theta)L(0,\mu_2^{-1}\theta^{-1})}{L(0,\chi)}.$$

By Hida and Finis there exists a character θ such that the numerator is a p-adic unit. From this we deduce that $\operatorname{Eis}^\theta\omega_\chi$ needs to be multiplied by at least $L^{\mathrm{alg}}(0,\chi)$ to make it integral, and hence get the lower bound on the denominator of $\operatorname{Eis}\omega_\chi$. This "twisting" technique was probably first used by C. Kaiser in the context of $\mathrm{GL}_{2/\mathbf{Q}}$ in [Ka] and was rediscovered in [S02a].

The third part of the proof of Theorem 1.1 is to show that there exists an integral class with the same restriction to the boundary as $\operatorname{Eis}\omega_\chi$. If $H^2_c(S,\mathcal{O}_\chi)_{\mathrm{torsion}}$ were trivial, this would not be a problem; unfortunately this is not the case, as shown in R. Taylor's thesis [T] and in calculations by Feldhusen [F]. We therefore need to understand the image of the restriction map to $H^1(\partial_S,\mathcal{O}_\chi)$ better. We achieve this upon demanding in addition that χ be unramified. Starting with a group cohomological result for $\mathrm{SL}_2(\mathcal{O})$ due to Serre [Se70] (which we extend to all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$, as suggested by [BN]) we define an involution on $H^1(\partial_S, \mathcal{O}_\chi)$ such that

$$H^1(S, \mathcal{O}_\chi) \overset{\text{res}}{\twoheadrightarrow} H^1(\partial_S, \mathcal{O}_\chi)^-,$$

where the superscript "-" indicates the -1-eigenspace. To prove this we generalize a theorem of Bianchi and carefully analyze the boundary of the Borel-Serre compactification of the adelic symmetric space before extending Serre's result. We apply the resulting criterion to $\mathrm{res}(\mathrm{Eis}\,\omega_\chi)$ to deduce the existence of a lift to $H^1(S, \mathcal{O}_\chi)$. From these three parts it is then not difficult to deduce Theorem 1.1.

In Chapter VII we apply Theorem 1.1 to get lower bounds for the order of Selmer groups. Let us denote by M the 1-dimensional p-adic Galois representation given by $\chi_p\epsilon$, where χ_p is the p-adic Galois character associated to χ and ϵ the p-adic cyclotomic character. Following Greenberg [G89] we associate a Selmer group $\mathrm{Sel}_F(M)$ to M. We prove the following proposition using techniques developed by Wiles, Urban, and Skinner (cf. [W86], [W90], [U01], [S04]).

Proposition 1.3. *Under the same assumptions as Theorem 1.1*

$$\mathrm{val}_p(\#\mathrm{Sel}_F(M)) \geq \mathrm{val}_p(\#(\mathbf{T}_\chi/\mathbf{I}_{\mu_1,\mu_2})).$$

Since Taylor [T2] associates Galois representations only to cuspidal automorphic forms on $\mathrm{GL}_2(\mathbf{A}_F)$ with cyclotomic central character we need to modify the cuspidal forms arising from the cohomological congruences. This is where the extra freedom in factoring our Hecke character $\chi = \mu_1/\mu_2$ comes in useful: any unramified anticyclotomic χ can be factored as $\chi = \mu \cdot \overline{\mu^c}$ such that this extra condition is satisfied (see Lemma 7.24). Note that this factorization is in general different from the one used in the application of the results of Hida and Finis. The translation between the two different factorizations is achieved by twisting the cuspforms and Galois representations.

Together with Theorem 1.1 the proposition immediately implies the bound for the Selmer group.

Theorem 1.4. *Let* $\chi : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ *be an unramified anticyclotomic Hecke character of infinity type* z^2*. Then*

$$\mathrm{val}_p(\#\mathrm{Sel}_F(M)) \geq \mathrm{val}_p(\#(\mathcal{O}_\chi/(L^{\mathrm{alg}}(0,\chi)))).$$

To conclude, we want to remark that the statement about these Selmer groups is also a consequence of the anticyclotomic Main Conjecture for imaginary quadratic fields, proved by Rubin in [Ru2] using Euler systems and by Tilouine in [Ti] using congruences between classical modular forms. Work on a "generalized Kummer's criterion" (with Selmer groups for finite order characters) for imaginary quadratic fields started with Coates and Wiles [CW] and Hida [Hi82]. However, the method presented here is very different; we construct elements in the Selmer groups and give lower bounds on their size. Our hope is that our methods generalize to higher rank groups.

CHAPTER II

Background

This chapter has two aims: to introduce notation and establish some conventions, and to list facts (mostly without proof) that will be used in later chapters.

2.1 Basic notation

Let F be an imaginary quadratic field and σ its nontrivial automorphism. For a place v of F let F_v be the completion of F at v. We write $\mathcal{O}$ for the ring of integers of F, $\mathcal{O}_v$ for the closure of $\mathcal{O}$ in F_v, and $\hat{\mathcal{O}}$ for $\prod_{v \text{ finite}} \mathcal{O}_v$. We fix once and for all an embedding $\overline{F} \hookrightarrow \overline{F}_v$ for each place v of F. For each prime $\mathfrak{p}$ we also fix an embedding $\overline{F}_\mathfrak{p} \hookrightarrow \mathbf{C}$ that is compatible with the fixed embeddings $\overline{F} \hookrightarrow \overline{F}_\mathfrak{p}$ and $\overline{F} \hookrightarrow \mathbf{C} (= \overline{F}_\infty)$. Complex conjugation is denoted by $z \mapsto \overline{z}$. We use the notations $\mathbf{A}, \mathbf{A}_f$ and $\mathbf{A}_F, \mathbf{A}_{F,f}$ for the adeles and finite adeles of $\mathbf{Q}$ and F, respectively, and write $\mathbf{A}^*$ and $\mathbf{A}_F^*$ for the group of ideles. For an $\mathcal{O}$-ideal $\mathfrak{m}$, define $U(\mathfrak{m}) = \prod_{v \text{ finite}} U_v(\mathfrak{m})$ with $U_v(\mathfrak{m}) = \{x \in \mathcal{O}_v : x \equiv 1 \mod \mathfrak{m}\mathcal{O}_v\}$. Also let $F_\mathbf{A}(\mathfrak{m}) = \{x \in \mathbf{A}_F^* : x_v = 1 \text{ if either } v \text{ is infinite or } \mathfrak{m}\mathcal{O}_v \neq \mathcal{O}_v\}$. Denote the class group of F by $\mathrm{Cl}(F)$ and the ray class group modulo $\mathfrak{m}$ by $\mathrm{Cl}_\mathfrak{m}(F)$. We write $\mathcal{D}$ for the different of F and $d_F = \mathrm{Nm}(\mathcal{D})$ for the (absolute) discriminant.

For any algebraic group $H/\mathbf{Q}$ and any ring A containing $\mathbf{Q}$ we write $H(A)$ for the group of A-valued points. We shall abbreviate $H_\infty = H(\mathbf{R})$.

2.2 The algebraic group

We denote by G the algebraic group $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_{2/F})$. The group $G_0/F = \mathrm{GL}_{2/F}$ has subgroups

$$B_0/F = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$$

$$U_0/F = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\}$$

$$T_0/F = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$$

$$Z_0/F = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} : \lambda \in \mathrm{GL}_{1/F} \right\},$$

the standard Borel subgroup, its unipotent radical, a maximal split torus, and the center of G_0/F, respectively. The restriction of scalars gives corresponding subgroups $B/\mathbf{Q}, T/\mathbf{Q}, U/\mathbf{Q}$ and $Z/\mathbf{Q}$ of G. We fix an isomorphism of $\mathbf{G}_m/F$ with the subtorus of T_0/F of elements of determinant 1, denoted by $T_0^{(1)}/F$, namely $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$.

We single out the element $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G(\mathbf{Q})$.

The positive simple root defines a homomorphism

$$\alpha_0 : B_0/F \to \mathbf{G}_m/F$$

$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto t_1/t_2$$

and we denote by α the corresponding homomorphism from $B/\mathbf{Q} \to \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m$. From [HaGL2] we take the notation $|\alpha|$ for $|\,| \circ \alpha_{\mathbf{A}} : B(\mathbf{A}) \to \mathbf{C}^*$, where $|\,| : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ is the idelic absolute value $x \mapsto |x| = \prod_v |x_v|_v$ Here we take the usual normalized absolute values for the local absolute values, except for the complex place, where we take $|x_\infty|_\infty = x_\infty \overline{x}_\infty$.

2.3 Symmetric spaces

In $G_\infty = G(\mathbf{R}) = G_0(\mathbf{R} \otimes F) = \mathrm{GL}_2(\mathbf{C})$ we choose the subgroup $K_\infty = U(2) \cdot Z_0(\mathbf{C}) = U(2) \cdot \mathbf{C}^*$ containing the maximal compact subgroup of unitary matrices. The symmetric space $X = G_\infty/K_\infty$ can be identified with the three-dimensional hyperbolic space $\mathbf{H}_3 = \mathbf{R}_{>0} \times \mathbf{C}$. One can view elements $(r, x+iy)$ of $\mathbf{H}_3$ with $r, x, y \in \mathbf{R}$, $r > 0$ as quaternions $q = x + yi + rj + 0 \cdot k$ for $1, i, j, k$ the standard $\mathbf{R}$-basis of the quaternions. Using this interpretation, the group $\mathrm{SL}_2(\mathbf{C})$ acts on $\mathbf{H}_3$ via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .q = (aq+b)(cq+d)^{-1},$$

where the inverse is taken in the skew field of quaternions. The $\mathrm{GL}_2(\mathbf{C})$-action on $\mathbf{H}_3$ is then given by $g.q := (\det(\mathrm{g})^{-1/2} g).q$. This action can be described geometrically as follows: An element $M \in \mathrm{SL}_2(\mathbf{C})$ acts via the usual fractional linear transformations on the Riemannian sphere $\mathbf{P}^1(\mathbf{C}) = \mathbf{C} \cup \{\infty\}$. The "Poincaré extension" of the action to $\mathbf{H}_3$, which we sketch below, agrees with the action described earlier (for details and references see [EGM] pp. 2/3). The biholomorphic map on $\mathbf{P}^1(\mathbf{C})$ induced by M may be represented as an even number of inversions in circles and reflections in lines in $\mathbf{C}$. Regarding $\mathbf{P}^1(\mathbf{C})$ as lying on the boundary of $\mathbf{H}_3$ as $r = 0$ there exists for each circle C and line L in $\mathbf{C}$ a unique (Euclidean) hemisphere $\hat{C}$ or plane $\hat{L}$ in $\mathbf{H}_3$ intersecting $\mathbf{P}^1(\mathbf{C})$ along the circle C or line L, respectively. The Poincaré extension to $\mathbf{H}_3$ of the action of M is the corresponding product of inversions in $\hat{C}$ and reflections in $\hat{L}$.

Arithmetic subgroups $\Gamma \subset \mathrm{GL}_2(F)$, i.e., subgroups commensurable to $\mathrm{GL}_2(\mathcal{O})$, act properly discontinously on $\mathbf{H}_3$; for torsion-free Γ the action is free and the quotient $\Gamma\backslash\mathbf{H}_3$ is a non-compact, complete, orientable Riemannian manifold of dimension 3. For any Γ there exists a torsion-free normal subgroup Γ' of finite index, so $\Gamma\backslash\mathbf{H}_3$ is the quotient of a differentiable manifold by a finite group, or a hyperbolic 3-orbifold (cf. [MR] Definition 1.3.3).

For any choice of an open compact subgroup $K_f \subset G(\mathbf{A}_f)$ we put $K = K_\infty K_f \subset G(\mathbf{A})$. For any algebraic group $H/\mathbf{Q}$ denote by K_f^H the intersection of K_f with $H(\mathbf{A}_f)$, by K_∞^H the intersection $K_\infty \cap H_\infty$. We write K_f^0 for the maximal compact subgroup $\mathrm{GL}_2(\widehat{\mathcal{O}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \widehat{\mathcal{O}}, ad - bd \in \widehat{\mathcal{O}}^* \right\}$. We will deal with the following congruence subgroups: For an ideal $\mathfrak{N}$ in $\mathcal{O}_F$ and a finite place v of F let $\mathfrak{N}_v = \mathfrak{N}\mathcal{O}_v$. We then put

$$K^1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_f^0, a - 1, c \equiv 0 \mod \mathfrak{N} \right\}$$

and

$$K^1(\mathfrak{N}_v) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_v), a - 1, c \equiv 0 \mod \mathfrak{N}_v \right\}$$

For calculations with Hecke operators it will be more convenient to deal with adelic symmetric spaces. For any choice of an open compact subgroup $K_f \subset G(\mathbf{A}_f)$ we define the space

$$S_{K_f} = G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K_f.$$

These are, in fact, as topological spaces just a finite, disjoint union of locally symmetric spaces:

$$S_{K_f} \cong \coprod_{i \in I} \Gamma_i \backslash \mathbf{H}_3.$$

This follows from considering the determinant map

$$S_{K_f} \twoheadrightarrow H_K := F^* \backslash \mathbf{A}_F^* / \det(K_f) \mathbf{C}^*.$$

The idele class group on the right hand side is a finite set, and the fibers of this map are connected since strong approximation holds for $\mathrm{Res}_{F/\mathbf{Q}} \mathrm{SL}_{2/F}$. Any $\xi \in G(\mathbf{A}_f)$ gives rise to an injection $j_\xi : G_\infty \to G(\mathbf{A})$ with $j_\xi(g_\infty) = (g_\infty, \xi)$ and, after taking quotients, to a component $\Gamma_\xi \backslash G_\infty / K_\infty \to G(\mathbf{Q})\backslash G(\mathbf{A})/K$, where $\Gamma_\xi := G(\mathbf{Q}) \cap \xi K_f \xi^{-1}$. This component is the fiber over $\det(\xi)$.

2.4 Lie algebra

(References: [Ha79], [Ha82], [Ko]) The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G/\mathbf{Q})$ is a $\mathbf{Q}$-vector space and we define $\mathfrak{g}_\infty = \mathfrak{g} \otimes_{\mathbf{Q}} \mathbf{R}$. Then $\mathfrak{g}_\infty$ is the Lie algebra of the real group $G_\infty = G(\mathbf{R}) = \mathrm{GL}_2(\mathbf{C})$ and so equals the two-by-two complex matrices $M_2(\mathbf{C})$ thought of as an $\mathbf{R}$-vector space. It carries a positive semidefinite K_∞-invariant form, the Killing form

$$\langle X, Y\rangle = \frac{1}{16}\mathrm{trace}(\mathrm{ad}X \cdot \mathrm{ad}Y),$$

and with respect to this form we have an orthogonal decomposition $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{p}$, where $\mathfrak{k}_\infty = \mathrm{Lie}(K_\infty)$ and

$$\mathfrak{p} = \mathbf{R}H \oplus \mathbf{R}E_1 \oplus \mathbf{R}E_2 := \mathbf{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

The group K_∞ acts on $\mathfrak{p}$ by the adjoint action. Let $P/\mathbf{Q}$ be any Borel subgroup of G. Under the action of K_∞^P we have a canonical decomposition of $\mathfrak{p} = \mathfrak{p}_{0,P} \oplus \mathfrak{p}_{1,P}$, where $\mathfrak{p}_{0,P}$ is the 1-dimensional subspace on which K_∞^P acts trivially and $\mathfrak{p}_{1,P}$ is 2-dimensional and an irreducible K_∞^P-module. In the case $P = B$ this decomposition becomes $\mathfrak{p} = \mathbf{R}H \oplus (\mathbf{R}E_1 \oplus \mathbf{R}E_2)$.

Let $S_\pm := \frac{1}{2}(\pm E_1 \otimes_{\mathbf{R}} 1 - E_2 \otimes_{\mathbf{R}} i) \in \mathfrak{p}_{\mathbf{C}}$ and denote by $\check{S}_\pm$ the dual vectors with respect to the Killing form.

The adjoint action of $k_\infty = \begin{pmatrix} \alpha & \overline{\beta} \\ -\beta & \overline{\alpha} \end{pmatrix} \in \mathrm{SU}_2(\mathbf{C}) \subset K_\infty$ on $\mathfrak{p}_{\mathbf{C}}$ is given by

$$k_\infty . \begin{pmatrix} S_+ \\ H \\ S_- \end{pmatrix} = \begin{pmatrix} \alpha^2 & \alpha\beta & \beta^2 \\ -2\alpha\overline{\beta} & \alpha\overline{\alpha} - \beta\overline{\beta} & 2\overline{\alpha}\beta \\ \overline{\beta}^2 & -\overline{\alpha}\overline{\beta} & \overline{\alpha}^2 \end{pmatrix} \begin{pmatrix} S_+ \\ H \\ S_- \end{pmatrix}. \tag{2.1}$$

In the center of the enveloping algebra of $\mathfrak{g}_\infty \otimes_{\mathbf{R}} \mathbf{C}$ we have the two Casimir operators $D' = X'Y' + Y'X' + H'^2/2$ and $D'' = X''Y'' + Y''X'' + H''^2/2$, where $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, where $A'.f = (\partial/\partial z)_{z=0}((1 + zA).f)$ and $A''.f = (\partial/\partial \overline{z})_{z=0}((1 + zA).f)$ for each matrix A of $\mathfrak{g}_\infty$.

Denote the Lie algebra over $\mathbf{Q}$ corresponding to T by $\mathfrak{t}$, the Lie algebra over F corresponding to T_0 by $\mathfrak{t}_0$, and the Lie algebra of $T(\mathbf{R})$ by $\mathfrak{t}_\infty$. Similarly use $\mathfrak{u}$ and $\mathfrak{b}$ for the Lie algebras of U and B with the same convention on subscripts.

2.5 Modules

In this section we gather some facts about modules of the group G. We follow the notation of [F] pp.9-10, 12-13, and [Ko] pp. §1.2.

The group $\mathrm{GL}_2(F)$ acts on the F-vector space $M^n := \mathrm{Sym}^n(F^2)$ of homogeneous polynomials of degree n in two variables X and Y with coefficients in F by right translation (coordinatized version of the n-th symmetric power of the standard representation on F^2):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .X^iY^{n-i} = (aX+bY)^i(cX+dY)^{n-i}. \tag{2.2}$$

Applying first the field automorphism σ to the entries a, b, c and d, we get another representation $\overline{M}^n$. We also have one-dimensional representations $F[k,l]$ for $(k,l) \in \mathbf{Z}^2$, on which $g \in G$ acts by multiplication by $\det^k(g) \cdot \sigma(\det(g))^l$. We obtain the representations $M^{m,n}[k,l] := M^m \otimes_F \overline{M}^n \otimes_F F[k,l]$, $M^m[k] := M^m \otimes_F F[k,0]$, and $\overline{M}^n[l] := \overline{M}^n \otimes_F F[0,l]$.

There is an isomorphism of $\mathrm{GL}_2(F)$-modules $M^{m,n} := M^{m,n}[0,0]$ and its F-dual $(M^{m,n})^\vee$ induced by the pairing

$$\langle\, ,\, \rangle : M^{m,n} \times M^{m,n} \to F,$$

$$X^jY^{m-j}\overline{X}^k\overline{Y}^{n-k} \times X^\mu Y^{m-\mu}\overline{X}^\nu\overline{Y}^{n-\nu} \mapsto (-1)^{j+k} \begin{pmatrix} m \\ j \end{pmatrix}^{-1} \begin{pmatrix} n \\ k \end{pmatrix}^{-1} \delta_{j,m-\mu}\delta_{k,n-\nu}.$$

This is the coordinatized version of the pairing induced by the determinant pairing on F^2 (cf.[Hi93] p. 169).

We note that in each F-vector space $M^{m,n}[k,l]$ the $\mathcal{O}$-lattice of polynomials with $\mathcal{O}$-coefficients is stable under the arithmetic subgroup $\mathrm{GL}_2(\mathcal{O})$.

For an $\mathcal{O}[G(\mathbf{Q})]$-module M we denote $M \otimes_{\mathcal{O}} A$ by M_A for any $\mathcal{O}$-algebra A. Note that for $M = M^m$ one has an action of $M_2(A)$ on M_A given by (2.2).

2.6 Hecke characters

A Hecke character (or Größencharakter) of F is a continuous group homomorphism $\lambda : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ and decomposes as a product of local characters $\lambda = \prod_v \lambda_v$. The largest ideal $\mathfrak{m}$ such that λ is trivial on $U(\mathfrak{m})$ is called the conductor of λ and denoted by $\mathfrak{f}_\lambda$. A Hecke character λ corresponds uniquely to a character on ideals prime to the conductor (see [Hi93] §8.2). Under this correspondence $\lambda(\mathfrak{a})$ for an ideal $\mathfrak{a}$ equals $\lambda(a)$ for any finite idele $a \in F_{\mathbf{A}}(\mathfrak{f}_\lambda)$ such that the fractional ideal corresponding to a equals $\mathfrak{a}$.

The archimedean part $\lambda_\infty : \mathbf{C}^* \to \mathbf{C}^*$ is of the form $z \mapsto \frac{z^a\bar{z}^b}{(z\bar{z})^t}$ for $t \in \mathbf{C}, a, b \in \mathbf{Z}$. We will say that λ has infinity type $\frac{z^a\bar{z}^b}{(z\bar{z})^t}$. If S_λ denotes the set of finite places of F which are ramified for λ (i.e. those that divide $\mathfrak{f}_\lambda$), we define the (incomplete) L-series $L(s, \lambda)$ by the Euler product

$$L(s, \lambda) := \prod_{v \notin S_\lambda} (1 - \lambda(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v)^{-s})^{-1},$$

where $\mathfrak{P}_v$ is the maximal ideal in $\mathcal{O}_v$.

The L-series $L(s, \lambda)$ can be continued to a meromorphic function on the whole complex plane and satisfies a functional equation, which is proven, for example, in [Hi93] §8.6 or [La] XIV Theorem 14. We state here the functional equation for unitary characters λ of infinity type $\frac{z^m}{(z\bar{z})^{m/2}}$ for $m \in \mathbf{Z}$:

Define the completed L-function by

$$\Lambda(s, \lambda) := \left(\frac{2\pi}{\sqrt{\mathrm{Nm}(\mathcal{D}\mathfrak{f}_\lambda)}}\right)^{-s} \Gamma\left(s + \frac{|m|}{2}\right) L(s, \lambda),$$

with $\mathfrak{f}_\lambda$ the conductor of λ and $\mathcal{D}$ the different of F.

Then the functional equation is

$$W(\lambda)\Lambda(s, \lambda) = \Lambda(1 - s, \overline{\lambda}),$$

where the root number $W(\lambda)$ is of absolute value 1 and given by

$$W(\lambda) = i^{-m}(\mathrm{Nm}(\mathfrak{f}_\lambda))^{-1/2} \prod_{v \in S_\lambda} \tau_v(\lambda) \prod_{v \notin S_\lambda} \lambda(\mathcal{D}_v^{-1}).$$

Here the Gauss sum τ_v is given by

$$\tau_v(\lambda_v) = \sum_{\epsilon\in\mathcal{O}_v^*/(1+f_{\lambda,v})} (\lambda\mathbf{e})(\epsilon\pi^{-\mathrm{ord}_v(\mathfrak{f}_\lambda\mathcal{D})})$$

for $\mathbf{e}$ the standard additive character $\mathbf{e}(x) = \exp(-2\pi i[\mathrm{tr}_{F_v/\mathbf{Q}_l}]_\ell)$, where ℓ is the residual characteristic of v and $[x]_\ell$ denotes the ℓ-fraction part for $x \in \mathbf{Q}_\ell$.

If the infinity type of λ is z^n, the functional equation takes the following form: The completed L-series is now

$$\Lambda(s,\lambda) := \left(\frac{2\pi}{\sqrt{\mathrm{Nm}(\mathcal{D}\mathfrak{f}_\lambda)}}\right)^{-s} \Gamma(s)L(s,\lambda).$$

Denoting by $\tilde{\lambda}$ the unitary character $\lambda|\cdot|^{-n/2}$ we get

$$W(\tilde{\lambda})\Lambda(1-n-s,\lambda) = \Lambda(s,\overline{\lambda}). \tag{2.3}$$

Define the character λ^c by $\lambda^c(x) = \lambda(\sigma(x))$. Since σ just permutes the Euler factors we have $L(s,\lambda) = L(s,\lambda^c)$.

We will use the following result of Shimura, Katz, Hida and Tilouine about the special L-value for $s = 0$:

Theorem 2.1. *Let p be a rational prime that splits in F, $\mathfrak{p}$ one of the prime ideals in F lying above it, and λ an algebraic Hecke character with conductor prime to p and of infinity type $z^k\left(\frac{z}{\bar{z}}\right)^\ell$, where k and ℓ are integers satisfying either $k > 0$ and $\ell \geq 0$ or $k \leq 1$ and $\ell \geq 1-k$. Then there exists a complex period Ω, independent of λ, such that*

$$L^{\mathrm{alg}}(0,\lambda) := \Omega^{-k-2\ell}\left(\frac{2\pi}{\sqrt{d_F}}\right)^\ell \Gamma(k+\ell)(1-\lambda(\overline{\mathfrak{p}}))(1-\lambda^*(\overline{\mathfrak{p}}))L(0,\lambda) \in \mathfrak{O},$$

where $\mathfrak{O}$ is the integer ring of an unramified finite extension of $F_\mathfrak{p}$, $d_F = \mathrm{Nm}(\mathcal{D})$ is the absolute discriminant of F and $\lambda^*(\overline{\mathfrak{p}}) = \lambda(\mathfrak{p})^{-1}\mathrm{Nm}(\mathfrak{p})$.

References. Shimura showed that this normalization is algebraic. Together, [K76] Chapters 4 and 8, [K78] Theorem 5.3.0, and [HT] Theorem II show that it is a p-adic integer in $\widehat{\overline{F}}_\mathfrak{p}$. With our fixed embedding $\overline{F} \hookrightarrow \overline{F}_\mathfrak{p}$ this shows that the value lies in a finite extension of $F_\mathfrak{p}$ and is p-integral. See also [Hi04a] Theorem 1.1 and [dS] II Theorem 4.12 and 4.14. □

We will be working with Hecke characters λ of type (A_0), i.e., characters with infinity type $z^a\overline{z}^b$ with $a, b \in \mathbf{Z}$ (cf. [We55]). For such characters $\mathbf{Q}(\mathrm{Im}(\lambda_f))$ is a number field and one can attach to λ a p-adic Galois character of $\mathrm{Gal}(\overline{F}/F)$, its p-adic avatar (cf. [Hi82] p. 248):

Let $\mathfrak{m}$ be the conductor of λ and let K be the finite extension of F containing $\lambda(x)$ for all $x \in F_{\mathbf{A}}(\mathfrak{m})$ and all conjugates of F over $\mathbf{Q}$. Fix a finite place v of K and write p for its residual characteristic. For $x \in F_p^* = (F \otimes_{\mathbf{Q}} \mathbf{Q}_p)^* = \prod_{w|p} F_w^*$ define $\lambda_\infty(x)$ using the extensions of the embeddings $F \hookrightarrow K$ to $F_w \hookrightarrow K_v$ and λ_∞. Now let $\lambda_v : F^*\backslash\mathbf{A}_F^*/U(\mathfrak{m})^{(p)}F_\infty^* \to K_v^*$ be the unique continuous character such that $\lambda_v(x) = \lambda(x)$ if $x \in F_{\mathbf{A}}(\mathfrak{m}p)$ and $\lambda_v(x_p) = \lambda(x_p)\lambda_\infty(x_p)$ for all $x_p \in F_p^*$. Using the Artin reciprocity map of class field theory, this gives rise to a Galois character $\lambda_{\mathfrak{p}} : \mathrm{Gal}(F(\mathfrak{m}p^\infty)/F) \to K_v^*$, where $F(\mathfrak{m}p^\infty)$ denotes the ray class field of conductor $\mathfrak{m}p^\infty$ and $\mathfrak{p}$ is the prime of F lying below v.

2.7 Automorphic forms

We want to use congruences between modular forms over imaginary quadratic fields. There are various ways of thinking of these:

- real analytic functions on hyperbolic three-space $\mathbf{H}_3$ (Maass forms)
- automorphic representations of $\mathrm{GL}_2(\mathbf{A}_F)$
- certain cohomology classes in the cohomology of quotients of $\mathbf{H}_3$ by congruence subgroups $\Gamma \subset \mathrm{GL}_2(\mathcal{O})$.

They are best susceptible to computation in the latter incarnation, and we will mainly be handling them in this form, but since we later want to work with automorphic forms and representations, we will recall their definition here (following the description in [U95], [U98]):

For a compact open subgroup $K_f \subset G(\mathbf{A}_f)$, we call a function $f : G(\mathbf{A}) \to M_{\mathbf{C}}^{2n+2}$ an automorphic form for K_f of weight $n \geq 0$, if it satisfies conditions (1)-(7) in the following list. If it also satisfies (8), we call it a cuspidal automorphic form.

(1) $f(\gamma g) = f(g)$ for all $\gamma \in G(\mathbf{Q})$

(2) $f(gz_\infty) = f(g) \cdot |z_\infty|_\infty^{-n}$ for all $z_\infty \in Z_0(\mathbf{C})$

(3) $f(gk_\infty) = k_\infty.f(g)$ for all $k_\infty \in U(2)$

(4) $f(gk) = f(g)$ for all $k \in K_f$

(5) With respect to any $U_2(\mathbf{C})$-invariant norm on $M_{\mathbf{C}}^{2n+2}$, $f \times |\det|_{\mathbf{A}_F}^n$ is square integrable on $S_{K_f}^G$.

(6) f is C^∞ and of moderate growth in its archimedean component

(7) f is an eigenvector of the Casimir operators D' and D' with eigenvalue $n + n^2/2$ (see [We71], pp. 67-68).

(8) For each $g \in G(\mathbf{A})$, one has

$$\int_{U(\mathbf{Q})\backslash U(\mathbf{A})} f(ug)du = 0.$$

We will denote the space of automorphic forms for K_f of weight n by $M_n(K_f, \mathbf{C})$ and the subspace of cuspidal forms by $S_n(K_f, \mathbf{C})$. For each continuous $\omega : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$, we further denote by $S_n(K_f, \omega, \mathbf{C})$ the space of forms satisfying in addition $f(gz) = f(g)\omega(z)$ for each $z \in Z(\mathbf{A})$.

$G(\mathbf{A})$ acts on these spaces by right translation. As explained in [U95] §3.1,

$$S_n(K_f, \mathbf{C}) \cong \bigoplus_{\Pi} V_{\Pi_f}^{K_f}, \tag{2.4}$$

where the sum is over cuspidal automorphic representations Π with Π_∞ isomorphic to the principal series representation of $\mathrm{GL}_2(\mathbf{C})$ corresponding to the character

$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \left(\frac{t_1}{|t_1|_\infty^{1/2}}\right)^{n+1} \left(\frac{|t_2|_\infty^{1/2}}{t_2}\right)^{n+1} \cdot |t_1 t_2|_\infty^{-n/2}.$$

For $S_n(K_f, \omega, \mathbf{C})$ one has a similar decomposition restricting above sum to the cuspidal automorphic representations with central character ω (to be defined later). For the exact definition of cuspidal automorphic representations we refer to [Gel]. We recall here only that a cuspidal automorphic representation Π factors as $\Pi = \Pi_\infty \otimes \Pi_f$

for Π_∞ a representation of $\mathrm{GL}_2(\mathbf{C})$ and Π_f an irreducible representation of $\mathrm{GL}_2(\mathbf{A}_f)$ whose space we denote by V_{Π_f}. The multiplicity one theorem of Jacquet and Langlands says that each isomorphism class of automorphic representations occurs only once in this decomposition. The $\mathrm{GL}_2(\mathbf{A}_f)$-representation Π_f further factors as $\Pi_f = \bigotimes_v \Pi_v$ with each Π_v an irreducible admissible representation of $\mathrm{GL}_2(F_v)$. All but finitely many Π_v ($v \notin S$ for some finite set of places S) are unramified, i.e. V_{Π_v} has a nonzero vector fixed by $\mathrm{GL}_2(\mathcal{O}_v)$. A representation $\Pi_v : \mathrm{GL}_2(F_v) \to \mathrm{GL}(V)$ for a complex vector space V is called admissible if (i) every vector $v \in V$ is fixed by some open subgroup of $\mathrm{GL}_2(F_v)$, and (ii) for every open compact subgroup K_v of $\mathrm{GL}_2(F_v)$ the subspace of vectors in V fixed by K_v is finite dimensional.

If K_v is an open compact subgroup of $\mathrm{GL}_2(F_v)$ we write V^{K_v} for the subspace of vectors fixed by K_v. For open compact subgroups K_v and K_v' and an element g of $\mathrm{GL}_2(F_v)$ we define the Hecke operator $[K_v g K_v'] : V^{K_v'} \to V^{K_v}$ by

$$[K_v g K_v']\phi = \sum_i h_i^{-1}.\phi$$

where $K_v g K_v' = \coprod_i K_v' h_i$. Similarly, for open compact subgroups K_f, K_f' of $\mathrm{GL}_2(\mathbf{A}_f)$ and $g \in \mathrm{GL}_2(\mathbf{A}_f)$, one has a Hecke action of $[K_f g K_f']$ on $V_{\Pi_f}^{K_f}$ and so, by (2.4), on $S_n(K_f, \mathbf{C})$. For $K_f = K_f' = K^1(\mathfrak{N})$ define $T_v = [K^1(\mathfrak{N}) \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} K^1(\mathfrak{N})]$ for all finite places v and $S_v = [K^1(\mathfrak{N}) \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} K^1(\mathfrak{N})]$ for places v not dividing $\mathfrak{N}$, where π_v is a uniformizer of F_v. If $\phi = \otimes_w \phi_w \in V_{\Pi_f}^{K^1(\mathfrak{N})}$ then T_v and S_v only act on ϕ_v. The operator T_v, for example, therefore has the same action as $[K^1(\mathfrak{N}_v) \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} K^1(\mathfrak{N}_v)]$. One can show that if Π_v is unramified (i.e., $v \notin S$) then $V_{\Pi_v}^{\mathrm{GL}_2(\mathcal{O}_v)}$ is 1-dimensional (see [Cas] and Section 3.2). This implies that any $\phi = \otimes_w \phi_w \in V_{\Pi_f}^{K_f}$ with $K_{f,v} = \mathrm{GL}_2(\mathcal{O}_v)$ has the same eigenvalue for T_v since ϕ_v is unique up to scalar. We will denote it by $a_v(\Pi)$.

Note also that by Schur's Lemma each Π has a central character $\omega = \otimes \omega_v$ with ω_v giving the action of the center of each $\mathrm{GL}_2(F_v)$. Note that if Π_v is unramified then $\omega_v(\pi_v)$ gives the inverse of the eigenvalue of Π_v for S_v.

2.8 Borel-Serre compactification

In general, the manifolds $\Gamma\backslash\mathbf{H}_3$ and S_{K_f} from Section 2.3 are not compact. There are several ways to compactify them, but the one most convenient for cohomological considerations is the Borel-Serre compactification (cf. [BS]). This compactification gives manifolds with corners and we will denote them by $\overline{\Gamma\backslash\mathbf{H}}_3$ and $\overline{S}_{K_f}$, respectively.

In fact, one first considers the Borel-Serre compactification $\overline{\mathbf{H}}_3$ of $\mathbf{H}_3$, a manifold with (countably many) corners. As a set, $\overline{\mathbf{H}}_3$ is given as union of $\mathbf{H}_3$ with boundary faces $e(P) = \mathbf{H}_3/A_P \cong U_P(\mathbf{R})$, one for each rational Borel subgroup P of G, where U_P denotes its unipotent radical and A_P the identity component of $P(\mathbf{R})/U_P(\mathbf{R})$, and the action of A_P on $\mathbf{H}_3$ is the geodesic action (cf. [BS]). The set is given a $G(\mathbf{Q})$-invariant topology.

In our situation this can be described very explicitely by viewing the boundary faces $e(P)$ as "horospheres minus a point" (see [BJ] III.5.15, [Ko] §1.4.4). This means that we view $\overline{\mathbf{H}}_3$ as

$$\overline{\mathbf{H}}_3 = \mathbf{H}_3 \cup \bigcup_{x\in\mathbf{P}^1(F)} (\mathbf{P}^1(\mathbf{C}) - \{x\}).$$

The group $G(\mathbf{Q})$ acts on $\bigcup_{x\in\mathbf{P}^1(F)}(\mathbf{P}^1(\mathbf{C}) - \{x\})$ by mapping $w \in \mathbf{P}^1(\mathbf{C}) - \{x\}$ to $\gamma w \in \mathbf{P}^1(\mathbf{C}) - \{\gamma x\}$ for $\gamma \in G(\mathbf{Q})$. Together with the usual action on $\mathbf{H}_3$ we therefore have defined an action of $G(\mathbf{Q})$ on $\overline{\mathbf{H}}_3$. We equip $\overline{\mathbf{H}}_3$ now with a $G(\mathbf{Q})$-invariant topology. On $\mathbf{H}_3$ we take the product topology of $(0,\infty) \times \mathbf{C}$, which agrees with the natural topology coming from $G_\infty/K_\infty \cong \mathbf{H}_3 = \mathbf{R}_{>0} \times \mathbf{C}$. For $x = [0:1](=\infty)$ we identify $\mathbf{P}^1(\mathbf{C}) - \{x\}$ with $\{\infty\} \times \mathbf{C}$ via $[1:z] \mapsto \{\infty\} \times \{z\}$. We then give $\mathbf{H}_3 \cup (\mathbf{P}^1(\mathbf{C}) - \{x\})$ the product topology of $(0,\infty] \times \mathbf{C}$. One checks that the group $B(\mathbf{Q})$ operates topologically on $\mathbf{H}_3 \cup (\mathbf{P}^1(\mathbf{C}) - \{x\})$. The topology on $\overline{\mathbf{H}}_3$ is then defined so that each $\gamma \in G(\mathbf{Q})$ acts as a topological automorphism. If we view the other cusps $[1:z] \in \mathbf{P}^1(F)$ as points $(0,z)$ in the complex plane $\{(0,c)|c \in \mathbf{C}\} \subset \mathbf{R}_{\geq 0} \times \mathbf{C}$ attached to $\mathbf{H}_3$ then neighborhoods of ∞ of the form $\{(r,z) : r \geq r_0\}$ correspond to Euclidean balls in $\mathbf{H}_3$ tangent to $\{0\} \times \mathbf{C}$ at $(0,z)$.

Such a ball of radius R can be identified with $(\mathbf{P}^1(\mathbf{C}) - \{[1:z]\}) \times (0, R]$ and the boundary face $\mathbf{P}^1(\mathbf{C}) - \{[1:z]\}$ is added as $(\mathbf{P}^1(\mathbf{C}) - \{[1:z]\}) \times \{0\}$. (This is where the terminology "horosphere" comes from, see also [MR] p. 54.)

Using reduction theory one shows that $\overline{\mathbf{H}}_3$ is Hausdorff. The natural inclusions of $\mathbf{H}_3$ and $e(P)$ into $\overline{\mathbf{H}}_3$ are embeddings of real manifolds. Moreover, $\mathbf{H}_3$ is open and each $e(P)$ is closed and the arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$ acts properly discontinuously on $\overline{\mathbf{H}}_3$ and $\overline{\mathbf{H}}_3 \setminus \mathbf{H}_3$. The quotient $\Gamma\backslash\overline{\mathbf{H}}_3$ is a Hausdorff compactification of $\Gamma\backslash\mathbf{H}_3$, also denoted by $\overline{\Gamma\backslash\mathbf{H}_3}$. Its boundary $\partial(\overline{\Gamma\backslash\mathbf{H}_3})$ is a finite union of tori $\Gamma_P\backslash e(P)$, with $\Gamma_P = \Gamma \cap P(\mathbf{Q})$, for a set of representatives of Γ-conjugacy classes of Borel subgroups (equivalently of $\mathbf{P}^1(F)/\Gamma$). Furthermore, $e'(P) := \Gamma_P\backslash e(P)$ is homotopy equivalent to $\Gamma_P\backslash\mathbf{H}_3$.

The Borel-Serre compactification of the adelic symmetric space is given by

$$\overline{S}_{K_f} = \coprod_{[\det(\xi)]\in H_K} \Gamma_\xi\backslash\overline{\mathbf{H}}_3 = \coprod_{[\det(\xi)]\in H_K} \Gamma_\xi\backslash\mathbf{H}_3 \sqcup_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\xi} \Gamma_{\xi,B^\eta}\backslash e(B^\eta), \tag{2.5}$$

with $H_K = \mathbf{A}_F^*/\det(K)F^*$, $\Gamma_\xi = G(\mathbf{Q}) \cap \xi K_f \xi^{-1}$ for $\xi \in G(\mathbf{A}_f)$, and $B^\eta(\mathbf{Q}) = \eta^{-1}B(\mathbf{Q})\eta$ for $\eta \in G(\mathbf{Q})$. The topology is such that $S_{K_f} \overset{i}{\hookrightarrow} \overline{S}_{K_f}$ is a homotopy equivalence.

For a very concise description for the Borel-Serre compactification of the adelic symmetric space agreeing with the compactification of its connected components $\Gamma_\xi\backslash\mathbf{H}_3$ sketched above, we refer to [HaGL2] §2.1. He shows that $\partial\overline{S}_{K_f}$ is homotopy equivalent to

$$\partial\tilde{S}_{K_f} := B(\mathbf{Q})\backslash G(\mathbf{Q})/K_f K_\infty \cong \coprod_{[\det(\xi)]\in H_K} \coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\xi} \Gamma_{\xi,B^\eta}\backslash\mathbf{H}_3,$$

where the boundary component $\Gamma_{\xi,B^\eta}\backslash\mathbf{H}_3$ gets embedded in $\partial\tilde{S}_{K_f}$ via $g_\infty \mapsto j_{\eta,\xi}(g_\infty) := \eta(g_\infty, \xi)$ (see [Ha82] p. 110). We note that together with the embeddings j_ξ defined in Section 2.3 we have the following commutative diagram:

$$\begin{array}{ccc} g_\infty\in\Gamma_{\xi,B^\eta}\backslash\mathbf{H}_3 & \xrightarrow{j_{\eta,\xi}} & B(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty = \partial\tilde{S}_{K_f} \\ \downarrow \qquad \downarrow & & \downarrow \text{projection} \\ g_\infty\in \Gamma_\xi\backslash\mathbf{H}_3 & \xrightarrow{j_\xi} & G(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty = S_{K_f} \end{array}$$

2.9 Cohomology of arithmetic groups

As mentioned above, we will be considering modular forms as cohomology classes in the cohomology of quotients of $\mathbf{H}_3$, or the adelic symmetric space S_{K_f}. We will use local coefficient systems associated with the $G(\mathbf{Q})$-modules $M^{m,n}$.

2.9.1 Sheaves

(a) Let us first consider the space $\Gamma\backslash\mathbf{H}_3$ for an arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$. Given an $\mathcal{O}[\Gamma]$-module N, we define an $\mathcal{O}$-module sheaf via its local sections for open $U \subset \Gamma\backslash\mathbf{H}_3$:

$$\widetilde{N}(U) := \{f : \pi_\Gamma^{-1}(U) \to N \text{ locally constant } : f(\gamma x) = \gamma.f(x) \forall x \in \pi_\Gamma^{-1}(U) \text{ and } \gamma \in \Gamma\}, \tag{2.6}$$

where $\pi_\Gamma : \mathbf{H}_3 \to \Gamma\backslash\mathbf{H}_3$ is the canonical projection.

For any $\mathcal{O}$-algebra R we similarly define an R-module sheaf $\widetilde{N_R}$. Note that this equals $\widetilde{N}_R := \widetilde{N} \otimes_{\underline{\mathcal{O}}} \underline{R}$, where we denote by $\underline{R}$ the constant sheaf associated to R.

(b) Similarly, we define for an $\mathcal{O}[G(\mathbf{Q})]$-module M the F-module sheaf $\widetilde{M_F}$ on S_{K_f} by

$$\widetilde{M_F}(U) := \{f : \pi^{-1}(U) \to M_F \text{ locally constant } | f(\gamma x) = \gamma.f(x) \forall x \in \pi^{-1}(U) \text{ and } \gamma \in G(\mathbf{Q})\}, \tag{2.7}$$

where $\pi : G(\mathbf{A})/K_\infty K_f \to S_{K_f}$ is the projection and $U \subset S_{K_f}$ is an open subset.

To define an integral structure on the cohomology groups $H^i(S_{K_f}, \cdot)$ we assume that there exists an $\mathcal{O}$-lattice M' in M_F such that $M'_{\hat{\mathcal{O}}} = M' \otimes \hat{\mathcal{O}}$ is stable under K_f (for $K_f \subset G(\hat{\mathbf{Z}})$ and $M = M^{m,n}[k,l]$ one can take M' to be the polynomials with $\mathcal{O}$-coefficients). This allows us to define an integral subsheaf $\widetilde{M_\mathcal{O}}$ of $\widetilde{M_F}$ (cf. [U98] §1.4, [Ko] §1.5, and [F] §1.2): For each open subset $U \subset S_{K_f}$ we let

$$\widetilde{M_\mathcal{O}}(U) := \{f \in \tilde{M}_F(U) : f(g) \in g_f M'_{\hat{\mathcal{O}}} \text{ for all } g \in \pi^{-1}(U)\}.$$

Clearly, $\widetilde{M_\mathcal{O}} \otimes \underline{F} = \widetilde{M_F}$. In general, we define $\widetilde{M_R}$ for any $\mathcal{O}$-algebra R as $\widetilde{M_\mathcal{O}} \otimes \underline{R}$. For $\xi \in G(\mathbf{A}_f)$ let $M_\xi := M_F \cap \xi.M'_{\hat{\mathcal{O}}}$. Then M_ξ is a locally free, finitely generated $\mathcal{O}$-

module with an action by $\Gamma_\xi = G(\mathbf{Q}) \cap \xi K_f \xi^{-1}$ and $j_\xi^*(\widetilde{M_\mathcal{O}}) \cong \widetilde{M_\xi}$ for $j_\xi : \Gamma_\xi\backslash\mathbf{H}_3 \hookrightarrow S_{K_f}$ from Section 2.3.

(c) Lastly, we define R-module sheaves $\widetilde{M_R}$ on $\partial\tilde{S}_{K_f} = B(\mathbf{Q})\backslash G(A)/K_f K_\infty$ as pullbacks of the corresponding sheaves on S_{K_f} via the canonical projection. In this case one has the relation $j_{\eta,\xi}^*(\widetilde{M_\mathcal{O}}) \cong \widetilde{M_\xi}$ for $j_{\eta,\xi} : \Gamma_{\xi,B^\eta}\backslash\mathbf{H}_3 \hookrightarrow \partial\tilde{S}_{K_f}$ from Section 2.8.

2.9.2 Sheaf cohomology and group cohomology

(a) For the definitions of sheaf cohomology we refer to Chapter 3 of [Hart], Chapter II of [B67], and [Go]. For a sheaf $\mathcal{F}$ on a topological space X, we denote by $H^i(X,\mathcal{F})$ (resp. $H^i_c(X,\mathcal{F})$) the i-th cohomology group of $\mathcal{F}$ (resp. with compact support), and the interior cohomology, i.e., the image of $H^i_c(X,\mathcal{F})$ in $H^i(X,\mathcal{F})$, by $H^i_!(X,\mathcal{F})$.

For the rest of this subsection let M be an $\mathcal{O}[G(\mathbf{Q})]$-module and R an $\mathcal{O}$-algebra. The R-modules $H^i(S_{K_f},\widetilde{M_R})$ are finitely generated. Since $S_{K_f} \stackrel{i}{\hookrightarrow} \overline{S}_{K_f}$ is a homotopy equivalence, we have a canonical isomorphism

$$H^i(S_{K_f},\widetilde{M_R}) \cong H^i(\overline{S}_{K_f}, i_*\widetilde{M_R})$$

and in what follows we will replace $i_*\widetilde{M_R}$ by $\widetilde{M_R}$ and also write $\widetilde{M_R}$ for the sheaf $j^*i_*\widetilde{M_R}$ on ∂S_{K_f}, for $j : \partial\overline{S}_{K_f} \hookrightarrow \overline{S}_{K_f}$. By [HaGL2] §2.1 we have

$$H^i(\partial\overline{S}_{K_f},\widetilde{M}) \cong H^i(\partial\tilde{S}_{K_f},\widetilde{M}).$$

The decomposition of the adelic symmetric space into connected components gives rise to canonical isomorphisms (see [Ko] §1.6 and [F] §1.2)

$$H^i(S_{K_f},\widetilde{M_R}) \cong \bigoplus_{[\det(\xi)]\in H_K} H^i(\Gamma_\xi\backslash\mathbf{H}_3, \widetilde{M_\xi}\otimes \underline{R})$$

and

$$H^i(\partial\tilde{S}_{K_f},\widetilde{M_R}) \cong \bigoplus_{[\det(\xi)]\in H_K} \bigoplus_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\xi} H^i(\Gamma_{\xi,B^\eta}\backslash\mathbf{H}_3, \widetilde{M_\xi}\otimes \underline{R}).$$

The above cohomology groups and isomorphisms are all functorial in R.

From the short exact sequence

$$0 \to i_! \widetilde{M}_R \to i_* \widetilde{M}_R \to i_* \widetilde{M}_R / i_! \widetilde{M}_R \to 0$$

and $i_* \widetilde{M}_R / i_! \widetilde{M}_R \cong j_*(j^* i_* \widetilde{M}_R)$ we get a long exact sequence (functorial in R)

(2.8)

$$\ldots \to H^1_c(S_{K_f}, \widetilde{M}_R) \to H^1(S_{K_f}, \widetilde{M}_R) \xrightarrow{\text{res}} H^1(\partial \overline{S}_{K_f}, \widetilde{M}_R) \xrightarrow{\partial} H^2_c(S_{K_f}, \widetilde{M}_R) \to \ldots$$

We have an operation of a Hecke algebra on the cohomology groups $H^i(S_{K_f}, \widetilde{M}_R)$ and $H^i(\partial \overline{S}_{K_f}, \widetilde{M}_R)$: For $x \in G(\mathbf{A}_f)$ such that $x \cdot M' \subset M'$ (where M' is an $\mathcal{O}$-lattice such that $M'_{\hat{\mathcal{O}}}$ is stable under K_f) one defines

$$[K_f x K_f] : H^i_?(S_{K_f}, \widetilde{M}_R) \to H^i_?(S_{K_f}, \widetilde{M}_R)$$

by

$$[K_f x K_f] = \text{tr}_{K_f, K_f \cap x K_f x^{-1}} \circ \text{r}_x \circ \text{res}_{K_f, K_f \cap x^{-1} K_f x},$$

where ? can be $\emptyset$ or c, r_x is the map from $H^i_?(S_{K_f \cap x^{-1} K_f x}, \widetilde{M}_R)$ to $H^i_?(S_{K_f \cap x K_f x^{-1}}, \widetilde{M}_R)$ induced by right multiplication by x^{-1}, tr denotes a transfer morphism and res a restriction map (the definition for $\partial \overline{S}_{K_f}$ is similar). We refer to [U98] §1.4.4 for the details. These actions are compatible with the restriction map res in the long exact sequence (2.8), and so we also get an action on $H^i_!(S_{K_f}, \widetilde{M}_R)$.

For $K_f = K^1(\mathfrak{N})$ we again single out the operators $T_v = [K^1(\mathfrak{N}) \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} K^1(\mathfrak{N})]$ and $S_v = [K^1(\mathfrak{N}) \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} K^1(\mathfrak{N})]$ (diamond operator, [U98] §1.4.5).

(b) If G is a group and A an abelian group with an action by G we denote by $H^i(G, A)$ the i-th cohomology group of G with coefficients in A. This is defined as the i-th right derived functor of the functor $A \mapsto A^G$. We consider the resolution of A given by the complex

$$0 \to A \xrightarrow{\epsilon} A^0(G, A) \xrightarrow{d_0} A^1(G, A) \xrightarrow{d_1} \ldots \xrightarrow{d_{i-1}} A^i(G, A) \xrightarrow{d_i} \ldots,$$

where $A^i(G, A) = \text{Maps}(G^{i+1}, A)$, $\epsilon(a)$ is the constant function equal to a on G and

$$d_i(f)(x_0, \ldots, x_{i+1}) = \sum_{j=0}^{i+1} (-1)^j f(x_0, \ldots, \widehat{x_j}, \ldots, x_{i+1}),$$

where the symbol '^' means that the variable under it should be omitted. Each $A^i(G,A)$ is a G-module by means of the action

$$(x.f)(x_0,\ldots,x_i) = x.f(x^{-1}\cdot x_0,\ldots,x^{-1}\cdot x_i)$$

for $x, x_0, \ldots x_i \in G$ (see [BW] IX §1). It follows from the arguments used in the proof of [Mil] Proposition II.4.13 that $0 \to A \to A^\bullet(G,A)$ is an acyclic resolution of A. Therefore the cohomology groups can be calculated as cohomology groups of the complex

$$A^0(G,A)^G \to \ldots A^i(G,A)^G \to \ldots$$

(see also [Mil] Propostion II.1.16). Elements of $A^i(G,A)^G = \mathrm{Hom}_G(G^{i+1},A)$ are called "homogeneous cochains". Since homogeneous cochains are determined uniquely by its restriction to systems of the form $(1, g_1, g_1g_2, \ldots, g_1\ldots g_i)$ one can also use the following "inhomogeneous cochains" to calculate the cohomology groups (see [Se79] VII). Let $F^i(G,A)$ be maps from G^i to A. Note that $A^i(G,A)^G$ is isomorphic to $F^{i-1}(G,A)$ via the map $f \mapsto f'$, where $f'(x_1,\ldots,x_i) = f(1, x_1, x_1x_2, \ldots, x_1\cdots x_i)$. Under this isomorphism d corresponds to the coboundary map

$$d_i'(f')(g_1,\ldots,g_{i+1}) = g_1.f'(g_2,\ldots,g_{i+1}) +$$
$$+\sum_{j=1}^{i}(-1)^j f'(g_1,\ldots,g_jg_{j+1},\ldots,g_{i+1}) + (-1)^{i+1}f'(g_1,\ldots,g_i).$$

The cohomology group $H^i(G,A)$ is then isomorphic to $\ker(d_i')/\mathrm{im}(d_{i-1}')$.

If A has the additional structure of an S-module for a commutative ring S and the G-action is S-linear, the above discussion carries over to the category of $S[G]$-modules and the group cohomology groups $H^i(G,A)$ are S-modules.

(c) For an arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$ and an $\mathcal{O}[\Gamma]$-module N we can in many cases relate the sheaf cohomology groups $H^i(\Gamma\backslash\mathbf{H}_3, \widetilde{N}_R)$ to the cohomology groups $H^i(\Gamma, N_R)$:

Proposition 2.2 ([HaCAG] Satz 2.9.1). *For $\mathcal{O}$-algebras R in which the orders of all finite subgroups of Γ are invertible there is a natural R-functorial isomorphism*

$$H^i(\Gamma\backslash\mathbf{H}_3, \widetilde{N}_R) \cong H^i(\Gamma, N_R).$$

Sketch of proof. We recall that the sheaf cohomology groups $H^i(\Gamma\backslash\mathbf{H}_3,\cdot)$ are defined as the right derived functors of the global section functor. We note that the functor $N_R \mapsto N_R^\Gamma$ used for the definition of group cohomology is the composite of $N_R \mapsto \widetilde{N}_R$ and $\widetilde{N}_R \mapsto H^0(\Gamma\backslash\mathbf{H}_3,\widetilde{N}_R)$. Under the assumption in the proposition the functor $N_R \mapsto \widetilde{N_R}$ is exact. In addition one can show that this functor takes injective $R[\Gamma]$-modules to acyclic R-module sheaves. It therefore maps an injective resolution of N_R to a resolution of $\widetilde{N}_R$ by acyclic sheaves. Taking global sections one gets a complex whose cohomology by definition gives the groups $H^1(\Gamma, N_R)$ but which is also naturally isomorphic to the cohomology groups $H^i(\Gamma\backslash\mathbf{H}_3,\widetilde{N}_R)$. □

Remark 2.3. A lemma in [F] shows that for any $\mathcal{O}$-algebra R, $R\otimes_{\mathcal{O}}\mathcal{O}[\frac{1}{6}]$ satisfies the conditions of the proposition for any arithmetic subgroup $\Gamma\subset G(\mathbf{Q})$.

2.9.3 Complex coefficient systems

From now on we further assume that M and N are finite-dimensional $\mathbf{C}$-vector spaces. We then have analytic tools to handle these cohomology groups. For a C^∞-manifold X denote by $\Omega^i(X)$ the space of $\mathbf{C}$-valued C^∞ differential i-forms with exterior derivative d^i, and by $\Omega^i(X,M) = \Omega^i(X)\otimes_{\mathbf{C}} M$ the space of M-valued smooth i-forms. Note that $\Omega^0(X,M) = C^\infty(X,M)$. We write Ω^i_X for the sheaf of C^∞ differential i-forms.

Proposition 2.4 (de Rham Theorem, [Hi93] Appendix Theorem 2). *For a locally constant sheaf $\mathcal{F}$ on X having values in the category of finite dimensional $\mathbf{C}$-vector spaces there is a natural isomorphism*

$$H^i((\Omega^\bullet_X\otimes_{\underline{\mathbf{C}}}\mathcal{F})(X); d^\bullet\otimes \mathrm{id}_{\mathcal{F}}) \cong H^i(X,\mathcal{F}).$$

Sketch of proof. By the Lemma of Poincaré (which states that the higher de Rham cohomology groups of the open unit disc in $\mathbf{C}^n$ all vanish) the complex formed by the sheaves $\Omega^\bullet_X\otimes_{\underline{\mathbf{C}}}\mathcal{F}$ provides a resolution of $\mathcal{F}$. Furthermore, one shows that the sheaves $\Omega^i_X\otimes_{\underline{\mathbf{C}}}\mathcal{F}$ are acyclic. The sheaf cohomology $H^i(X,\mathcal{F})$ is therefore naturally isomorphic to the cohomology of the complex obtained by taking the global sections. □

For $X = S_{K_f}$, $\partial\tilde{S}_{K_f}$, and $\Gamma\backslash\mathbf{H}_3$ we defined locally constant sheaves $\widetilde{M}$ and $\widetilde{N}$, respectively. In these cases we let $\Omega^i(X,\widetilde{M}) := (\Omega^i_X \otimes_{\underline{\mathbf{C}}} \widetilde{M})(X)$. De Rham's Theorem implies that

$$H^i(S_{K_f},\widetilde{M}) \cong H^i(\Omega^\bullet(S_{K_f},\widetilde{M})),$$
$$H^i(\partial\tilde{S}_{K_f},\widetilde{M}) \cong H^i(\Omega^\bullet(\partial\tilde{S}_{K_f},\widetilde{M})),$$

and

$$H^i(\Gamma\backslash\mathbf{H}_3,\widetilde{N}) \cong H^i(\Omega^\bullet(\Gamma\backslash\mathbf{H}_3,\widetilde{N})).$$

Note that

$$\Omega^i(\mathbf{H}_3,N)^\Gamma \cong \Omega^\bullet(\Gamma\backslash\mathbf{H}_3,\widetilde{N})$$

via $\omega \mapsto \omega\circ\pi$ for the canonical projection $\pi : \mathbf{H}_3 \to \Gamma\backslash\mathbf{H}_3$ (cf. [BW] VII §1). Similarly,

$$\Omega^i(S_{K_f},\widetilde{M}) \cong (\Omega^i(G(\mathbf{A})/K_fK_\infty)\otimes_{\mathbf{C}} M)^{G(\mathbf{Q})}$$

and

$$\Omega^i(\partial\tilde{S}_{K_f},\widetilde{M}) \cong (\Omega^i(G(\mathbf{A})/K_fK_\infty)\otimes_{\mathbf{C}} M)^{B(\mathbf{Q})}.$$

For $X = \Gamma\backslash\mathbf{H}_3$ the natural isomorphisms of Proposition 2.2 and 2.4 compose to give an isomorphism between de Rham cohomology and group cohomology. For future reference we want to state this isomorphism explicitly for degree 1:

Proposition 2.5. *The natural isomorphism*

$$H^1(\Omega^\bullet(\mathbf{H}_3,N)^\Gamma) \cong H^1(\Gamma\backslash\mathbf{H}_3,\widetilde{N}) \cong H^1(\Gamma,N)$$

is induced by any of the following maps on closed 1-forms: For a choice of basepoint $x_0 \in \mathbf{H}_3$ *assign to a closed 1-form* $\tilde{\omega}$ *with values in* N *the (inhomogeneous) 1-cocycle*

$$\mathcal{G}_{x_0}(\tilde{\omega}) : \gamma \mapsto \int_{x_0}^{\gamma.x_0} \tilde{\omega}.$$

Proof. This is well-known but since we cannot find a reference we give the argument here (see, however, [Co] Proof of Lemma 3.3.5.1 for a more general version). First one checks that $\mathcal{G}_{x_0}$ is well-defined. It is independent of the choice of path because

$d\tilde{\omega} = 0$. Also it is easy to check that the class of the cocycle is independent of the choice of x_0.

Since the functor $N \to N^\Gamma$ is left-exact and by Propositions 2.2 and 2.4 the functors $N \mapsto H^i(\Omega^\bullet(\mathbf{H}_3, N)^\Gamma)$ and $N \mapsto H^i(\Gamma, N)$ are erasable functors on $\mathbf{C}[\Gamma]$-modules (see [Hart] III.1 for the definition of erasable additive functors). This implies that both

$$(H^i(\Omega^\bullet(\mathbf{H}_3, \cdot)^\Gamma))_{i\geq 0} \text{ and } (H^i(\Gamma, \cdot))_{i\geq 0}$$

are universal δ-functors. (We again refer to [Hart] III.1 for the definition and properties of δ-functors.) By the universality of both δ-functors there is a unique sequence of isomorphisms $H^i(\Omega^\bullet(\mathbf{H}_3, \cdot)^\Gamma) \to H^i(\Gamma, \cdot)$ for each $i \geq 0$, starting with the canonical isomorphism in degree 0, which commute with the connecting homomorphism δ^i for each short exact sequence of $\mathbf{C}[\Gamma]$-modules. It suffices therefore to show that the map on closed 1-forms given above defines a morphism $H^1(\Omega^\bullet(\mathbf{H}_3, \cdot)^\Gamma) \to H^1(\Gamma, \cdot)$ extending the one in degree 0.

As we recalled above, $H^1(\Gamma, N)$ is calculated by taking Γ-invariants of the acyclic resolution $N \to A^\bullet(\Gamma, N)$ and computing the cohomology of the resulting complex. The de Rham cohomology group is calculated as the cohomology of the complex $N^\Gamma \to \Omega^\bullet(\Gamma\backslash\mathbf{H}_3, \widetilde{N}) = \Omega^\bullet(\mathbf{H}_3, N)^\Gamma$. This is the complex of Γ-invariants of the complex $N \to \Omega^\bullet(\mathbf{H}_3, N)$. Since $\mathbf{H}_3$ is contractible the Poincaré Lemma mentioned above in Proposition 2.4 implies that this latter complex is exact and therefore a resolution of N. Note also that both resolutions are functorial in N.

For any $x_0 \in \mathbf{H}_3$ the morphism $f^0 : C^\infty(\mathbf{H}_3, N) \to A^0(\Gamma, N)$ given by $\phi \mapsto (g \mapsto \phi(g.x_0))$ commutes with the maps from N (in each case taking an element $m \in N$ to the constant map equal to m):

$$\begin{array}{ccccccc}
N & \xrightarrow{\epsilon} & C^\infty(\mathbf{H}_3, N) & \xrightarrow{d^0} & \Omega^1(\mathbf{H}_3, N)^{d^1=0} & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \\
N & \xrightarrow{\epsilon} & A^0(\Gamma, N) & \xrightarrow{d_0} & A^1(\Gamma, N)^{d_1=0} & \longrightarrow & 0
\end{array}$$

To make the above diagram commute f^1 must take a closed 1-form $\tilde{\omega}$ to the homogeneous 1-cocycle

$$(\gamma_1, \gamma_2) \mapsto F(\gamma_2.x_0) - F(\gamma_1.x_0) = \int_{\gamma_1.x_0}^{\gamma_2.x_0} \tilde{\omega},$$

for $F \in C^\infty(\mathbf{H}_3, N)$ with $d^0(F) = \tilde{\omega}$. Applying Stokes's Theorem gives the expression as an integral. With the correspondence between homogeneous and inhomogeneous cocycles recalled above this is the map given in the statement of Proposition 2.5. After taking Γ-invariants of the resolutions f^0 and f^1 induce δ-functorial maps on the cohomology groups in degree 0 and 1 and so must be the canonical isomorphisms. □

The de Rham cohomology groups are also canonically isomorphic to relative Lie algebra cohomology groups. For the definition of the latter we refer to [BW] Chapter 1. The tangent space of $\mathbf{H}_3$ at the point $x^0 := 1K_\infty \in G_\infty/K_\infty$ can be canonically identified with $\mathfrak{g}_\infty/\mathfrak{k}_\infty$. For $g \in G_\infty$ let $\mathrm{L}_g : \mathbf{H}_3 \to \mathbf{H}_3$ be the left-translation by g and $\mathrm{D}_{\mathrm{L}_g}$ the differential of this map. Assume that the $G(\mathbf{Q})$-action on M extends to a representation of G_∞. Let $\omega_M : Z(\mathbf{R}) \to \mathbf{C}^*$ be the character describing the action on M and write $C^\infty(\Gamma\backslash\mathrm{GL}_2(\mathbf{C}))(\omega_M^{-1})$ for those functions in $C^\infty(\Gamma\backslash\mathrm{GL}_2(\mathbf{C}))$ on which translation by elements in $Z(\mathbf{R})$ acts via ω_M^{-1}.

We can then identify the $\mathbf{C}$-vector spaces

$$\Omega^i(\mathbf{H}_3, M)^\Gamma \cong \mathrm{Hom}_{K_\infty}(\Lambda^i(\mathfrak{g}_\infty/\mathfrak{k}_\infty), C^\infty(\Gamma\backslash\mathrm{GL}_2(\mathbf{C}))(\omega_M^{-1}) \otimes M),$$

by mapping an M-valued differential form $\tilde{\omega}$ to the $(\mathfrak{g}, K_\infty)$-cocycle ω given by $\omega(g)(\theta_1 \wedge \ldots \wedge \theta_i) := \tilde{\omega}(gK_\infty)(\mathrm{D}_{\mathrm{L}_g}(\theta_1), \ldots, \mathrm{D}_{\mathrm{L}_g}(\theta_i))$. The differentials of the complexes corresponds and we get (cf. [BW] VII Corollary 2.7)

$$H^i(\Gamma\backslash\mathbf{H}_3, \widetilde{M}) \cong H^i(\mathfrak{g}_\infty, K_\infty, C^\infty(\Gamma\backslash\mathrm{GL}_2(\mathbf{C}))(\omega_M^{-1}) \otimes M).$$

Similarly, one obtains

$$H^i(S_{K_f}, \widetilde{M}) \cong H^i(\mathfrak{g}_\infty, K_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_M^{-1}) \otimes M)$$

and

$$H^i(\partial\tilde{S}_{K_f}, \widetilde{M}) \cong H^i(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_M^{-1}) \otimes M).$$

The action of the Hecke operators on the Lie algebra cohomology groups can be described as follows: for $x \in G(\mathbf{A}_f)$ define an action of $K_f x K_f$ on

$$f \in C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$$

by

$$([K_f x K_f].f)(h) = \sum_i f(h x_i^{-1}),$$

where $K_f x K_f = \coprod_i K_f x_i$. The induced action on the Lie algebra cohomology corresponds to the one on sheaf cohomology $H^i(S_{K_f}, \widetilde{M})$ (cf. [S02a]).

The connection between cohomology and cuspidal automorphic forms is given by a generalization of the Eichler-Shimura isomorphism due to Harder:

Theorem 2.6. *For each compact open subgroup $K_f \subset G(\mathbf{A}_f)$ and for $n \geq 0$, one has canonical isomorphisms $\delta^i_{K_f}$:*

$$\delta^i_{K_f} : S_n(K_f, \mathbf{C}) \to H^i_!(S_{K_f}, \widetilde{M^{n,n}_{\mathbf{C}}}),\ i = 1, 2.$$

These isomorphisms are Hecke-equivariant.

Reference. [U98] Theorem 1.5.1 □

We also want to remark on the relationship between cuspidal cohomology and interior cohomology: Let

$$L^2_0(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)^\infty$$

be the subspace of

$$C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$$

of square integrable cuspidal functions (see [Schw] §1.6 and [HaGL2] §3.1 for the exact definition). The inclusion

$$L^2_0(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)^\infty \hookrightarrow C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$$

induces a map on Lie algebra cohomology. Its image in $H^*(S_{K_f}, \widetilde{M})$ is called cuspidal cohomology and denoted by $H^*_{\text{cusp}}(S_{K_f}, \widetilde{M})$.

Lemma 2.7.

$$H^*_{\text{cusp}}(S_{K_f}, \widetilde{M}) = H^*_!(S_{K_f}, \widetilde{M}).$$

References. For $\dim_{\mathbf{C}} M > 1$ this is the case for all number fields F by [HaGL2] (3.2.5). For F imaginary quadratic and $\dim_{\mathbf{C}} M = 1$ see [F] Proof of Satz §1.5 and [U95] Proof of Theorem 3.2. □

2.10 Eisenstein cohomology

The short exact sequence

$$0 \to H^1_!(S_{K_f}, \widetilde{M}_{\mathcal{O}}) \to H^1(S_{K_f}, \widetilde{M}_{\mathcal{O}}) \stackrel{\text{res}}{\to} \text{Im}(\text{res}) \to 0$$

splits Hecke-equivariantly after tensoring by $\mathbf{C}$; using Eisenstein series Harder constructed a section to the restriction map.

Remark 2.8. This gives a direct sum decomposition

$$H^1(S_{K_f}, \widetilde{M}_{\mathbf{C}}) = H^1_{\text{Eis}} \oplus H^1_!(S_{K_f}, \widetilde{M}_{\mathbf{C}})$$

for H^1_{Eis} the image of this section. By the "Manin-Drinfeld" principle (comparison of Hecke eigenvalues) the sequence already splits for F-modules. However, we will later try to exploit that in general it does not split for $\mathcal{O}$-coefficient systems. We will look for congruences between classes in the interior cohomology $\tilde{H}^1_!(S_{K_f}, \widetilde{M}_{\mathcal{O}})$ and the Eisenstein part $H^1_{\text{Eis}} \cap \tilde{H}^1(S_{K_f}, \widetilde{M}_{\mathcal{O}})$. Here '$\sim$' denotes the torsion-free parts of the cohomology groups.

We will in the following describe how Harder obtains a section to the restriction map. For this we first need to give a description of the boundary cohomology in terms of representations induced from algebraic Hecke characters (we state here only the description for the cohomology of degree 1, for the other cases and proofs see [HaGL2] and [F]):

2.10.1 Boundary cohomology

The set of characters of $T(\mathbf{A})$ which contribute to the boundary cohomology $H^1(\partial \overline{S}_{K_f}, \widetilde{M})$ depends on the $G(\mathbf{Q})$-representation M. Working with $M = M^{m,n}[k,l]$

we will say, in analogy to [Ha82] §4, that a character $\phi = (\mu_1, \mu_2) : F^*\backslash\mathbf{A}_F^* \times F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ is in $S_1(m,n,k,l)$ if its infinite component is

$$\mu_{1,\infty}(z) = z^{1-k}\overline{z}^{-n-l} \text{ and } \mu_{2,\infty}(z) = z^{-m-k-1}\overline{z}^{-l}$$

and in $\overline{S}_1(m,n,k,l)$ if

$$\mu_{1,\infty}(z) = z^{-m-k}\overline{z}^{1-l} \text{ and } \mu_{2,\infty}(z) = z^{-k}\overline{z}^{-n-l-1}.$$

(Note that for $m = n$ and $k = l$ complex conjugation interchanges S_1 and $\overline{S}_1$. This is the case we will be specializing to later.) The two types get swapped by the action of the Weyl group, where we define $w_0.\phi = |\alpha|\phi^{w_0} = (\mu_2 \cdot |\,|, \mu_1 \cdot |\,|^{-1})$, so we are in the so-called "balanced case" (cf. [HaGL2] §2.9)

For a character $\phi : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ we define the induced module

(2.9)
$$V_{\phi_f}^{K_f} = \{\Psi : G(\mathbf{A}_f) \to \overline{F} \,|\Psi(bg) = \phi_f(b)\Psi(g), \Psi(gk) = \Psi(g)\; \forall b \in B(\mathbf{A}_f), k \in K_f\}.$$

We use here the following convention: for any $\mathbf{Q}$-algebra R we consider characters ϕ of $T(R)$ also as characters of $B(R)$ by defining $\phi(b) := \phi(t)$ if $b = tu$ for $t \in T(R)$ and $u \in U(R)$.

Similarly, we let

$$V_{\phi,\mathbf{C}}^{K_f} = \left\{ \Psi : G(\mathbf{A}) \to \mathbf{C} \;\middle|\; \begin{array}{l} \Psi(bg) = \phi(b)\Psi(g), \Psi(gk) = \Psi(g)\; \forall b \in B(\mathbf{A}), k \in K_f, \\ \Psi \text{ is } K^\infty\text{-finite on the right} \end{array} \right\}.$$

Remark 2.9. This definition follows the one used in Harder's work. This is not the usual unitary induction and explains the discrepancy between the infinity types above and those of the cuspidal automorphic representations in Section 2.7.

The non-unitarily induced module $V_{\phi_f}^{K_f}$ is the same as that from the unitary induction of the character $\eta = (\eta_1, \eta_2) = (\mu_1, \mu_2)|\alpha|^{-1/2}$ with $\alpha : B(\mathbf{A}) \to \mathbf{C}^*$ as in Section 2.2.

The infinity types translate as follows in the case $m = n$ and $k = l = 0$ (because of our use of modular symbols this is the case of interest later on):

- If $\phi = (\mu_1, \mu_2) \in S_1(m, m, 0, 0)$ then $\eta_{1,\infty}(z) = z\overline{z}^{-m}(z\overline{z})^{-1/2}$ and $\eta_{2,\infty}(z) = z^{-m-1}(z\overline{z})^{1/2}$.
- If $\phi = (\mu_1, \mu_2) \in \overline{S}_1(m, m, 0, 0)$ then $\eta_{1,\infty}(z) = z^{-m}\overline{z}(z\overline{z})^{-1/2}$ and $\eta_{2,\infty}(z) = \overline{z}^{-m-1}(z\overline{z})^{1/2}$.

In particular, this requires $\chi := \eta_1/\eta_2$ (which will be our main focus later on) to have infinity type

$$\chi_\infty(z) = \left(\frac{z}{\overline{z}}\right)^{m+1}.$$

From now on we will follow Harder and use non-unitary induction and the infinity types given above.

We will now see how these specific infinity types arise. For $\lambda : T(\mathbf{R}) \to \mathbf{C}^*$ consider the induced Harish-Chandra module

$$V_\lambda = \{f : G(\mathbf{R}) \to \mathbf{C} | f(bg) = \lambda(b)f(g) \text{ for } b \in B(\mathbf{R}), g \in G(\mathbf{R}), f \text{ is } K_\infty\text{-finite}\}.$$

Note that for $\phi : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ one has $V_{\phi,\mathbf{C}}^{K_f} = V_{\phi_\infty} \otimes_{\mathbf{C}} V_{\phi_f,\mathbf{C}}^{K_f}$. We decompose the Lie algebra

$$\mathfrak{g}_\infty = \mathfrak{k}_\infty + \mathfrak{b}_\infty = (\mathfrak{k}_\infty + \mathfrak{t}_\infty) \oplus \mathfrak{u}_\infty.$$

Then we get

$$\mathfrak{g}_\infty/\mathfrak{k}_\infty \cong \mathfrak{t}_\infty/(\mathfrak{t}_\infty \cap \mathfrak{k}_\infty) \oplus \mathfrak{u}_\infty$$

which is compatible with the action of K_∞^T on both sides. For any $G(\mathbf{Q})$-representation M evaluation at $1 \in G_\infty$ gives an identification

$$\mathrm{Hom}_{K_\infty}(\Lambda^i(\mathfrak{g}_\infty/\mathfrak{k}_\infty), V_\lambda \otimes M_{\mathbf{C}}) \cong \mathrm{Hom}_{K_\infty^T}(\Lambda^i(\mathfrak{t}_\infty/(\mathfrak{t}_\infty \cap \mathfrak{k}_\infty) \oplus \mathfrak{u}_\infty), \mathbf{C}\lambda \otimes M_{\mathbf{C}}),$$

where $\mathbf{C}\lambda$ denotes the 1-dimensional $T(\mathbf{R})$-module on which $T(\mathbf{R})$ acts by λ. The following formula is due to P. Delorme (see [HaGL2] p.68):

Lemma 2.10.

$$H^i(\mathfrak{g}_\infty, K_\infty, V_\lambda \otimes M_{\mathbf{C}}) = \bigoplus_{j=0}^{i} \mathrm{Hom}(\Lambda^{i-j}(\mathfrak{t}_\infty/\mathfrak{t}_\infty \cap \mathfrak{k}_\infty), (H^j(\mathfrak{u}_\infty, M_{\mathbf{C}}) \otimes \mathbf{C}\lambda)^{\mathfrak{t}_\infty}).$$

The Lie algebra cohomology groups $H^i(\mathfrak{u}_\infty, M_{\mathbf{C}}^{m,n})$ can be calculated by the following Lemma (cf. [HaGL2] §3.5, [F], Proof of §1.4 Satz):

Lemma 2.11. (a) *By the Künneth formula we have*

$$\begin{aligned} H^i(\mathfrak{u}_\infty, M_{\mathbf{C}}^{m,n}[k,l]) &= H^i(\mathfrak{u} \otimes_{\mathbf{Q}} \mathbf{C}, M_{\mathbf{C}}^{m,n}[k,l]) \\ &\cong \bigoplus_{j=0}^{i} \left(H^j(\mathfrak{u}_0 \otimes_F \mathbf{C}, M_{\mathbf{C}}^m[k]) \oplus H^{i-j}(\mathfrak{u}_0 \otimes_{F,\sigma} \mathbf{C}, \overline{M}_{\mathbf{C}}^n[l]) \right). \end{aligned}$$

(b)

$$H^i(\mathfrak{u}_0, M^m[k]) = \begin{cases} FX^m & \textit{if } i = 0, \\ FY^m \otimes U_\alpha^\vee & \textit{if } i = 1, \\ 0 & \textit{otherwise}, \end{cases}$$

where $U_\alpha^\vee$ *is a generator of* $\mathrm{Hom}(\mathfrak{u}_0, F)$*.*

We can therefore find cohomology classes $\tilde{e}^{(1,0)}$ and $\tilde{e}^{(0,1)}$ generating $H^1(\mathfrak{u}_0, M_{\mathbf{C}}^m[k]) \otimes H^0(\mathfrak{u}_0, \overline{M}_{\mathbf{C}}^n[l])$ and $H^0(\mathfrak{u}_0, M_{\mathbf{C}}^m[k]) \otimes H^1(\mathfrak{u}_0, \overline{M}_{\mathbf{C}}^n[l])$, respectively. They are eigenvectors for the action of the torus $T(\mathbf{R})$. Denoting the inverses of the eigencharacters by $\lambda_{1,0}(m,n,k,l)$ and $\lambda_{0,1}(m,n,k,l)$ respectively, we see that they are exactly the infinity types singled out above. By Delorme's formula $H^1(\mathfrak{g}_\infty, K_\infty, V_\lambda \otimes M_{\mathbf{C}}^{m,n})$ is nontrivial for $\lambda = \lambda_{1,0}(m,n,k,l)$ and $\lambda = \lambda_{0,1}(m,n,k,l)$.

We now have the following description of the cohomology of the boundary with complex coefficients. Recall that $H^1(\partial \overline{S}_{K_f}, \widetilde{M}_{\mathbf{C}}^{m,n})$ is isomorphic to

$$H^1(\partial \tilde{S}_{K_f}, \widetilde{M}_{\mathbf{C}}^{m,n}) \cong H^*(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_{M_{\mathbf{C}}^{m,n}}^{-1}) \otimes M_{\mathbf{C}}^{m,n}).$$

For each $\phi : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ in $S_1(m,n,0,0)$ or $\overline{S}_1(m,n,0,0)$ let

$$\Xi_\phi : H^*(\mathfrak{g}_\infty, K_\infty, V_{\phi,\mathbf{C}}^{K_f} \otimes M_{\mathbf{C}}^{m,n}) \to H^*(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_{M_{\mathbf{C}}^{m,n}}^{-1}) \otimes M_{\mathbf{C}}^{m,n})$$

be the map induced by the embedding $V_{\phi,\mathbf{C}}^{K_f} \hookrightarrow C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_{M_{\mathbf{C}}^{m,n}}^{-1})$. These are not injective. One can, however, show the following: For $(a,b) = (1,0)$ and $(0,1)$ let $[e^{(a,b)}]$ be generators of the 1-dimensional $\mathbf{C}$-vector spaces

$$\mathrm{Hom}(\Lambda^0(\mathfrak{t}_\infty/\mathfrak{t}_\infty \cap \mathfrak{k}_\infty), H^a(\mathfrak{u}_0, M_{\mathbf{C}}^m[k]) \otimes H^b(\mathfrak{u}_0, \overline{M}_{\mathbf{C}}^n[l]) \otimes \mathbf{C}\lambda_{a,b}(m,n,k,l))$$

represented by cocycles

$$\begin{aligned} e^{(a,b)} &\in \operatorname{Hom}_{K_\infty^T}(\Lambda^0(\mathfrak{t}_\infty/\mathfrak{t}_\infty\cap\mathfrak{k}_\infty)\otimes\mathfrak{u}_\infty, \mathbf{C}\lambda_{a,b}(m,n,k,l)\otimes M_{\mathbf{C}}^{m,n}) \\ &\subset \operatorname{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, V_{\lambda_{a,b}(m,n,k,l)}\otimes M_{\mathbf{C}}^{m,n}). \end{aligned}$$

Proposition 2.12. *We have an isomorphism of* $\mathbf{C}$*-vectorspaces*

$$\bigoplus_{\substack{\phi: T(\mathbf{Q})\backslash T(\mathbf{A})/K_f^T\to\mathbf{C}^*,\\ \phi\in S_1(m,n,k,l)}} \left(\mathbf{C}[e^{(1,0)}]\otimes V_{\phi_f}^{K_f}\right)\oplus\left(\mathbf{C}[e^{(0,1)}]\otimes V_{w_0.\phi_f}^{K_f}\right)$$

$$\xrightarrow{\sim} H^1(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega^{-1}_{M_{\mathbf{C}}^{m,n}})\otimes M_{\mathbf{C}}^{m,n})$$

given by (the restriction of) $\bigoplus_\phi(\Xi_\phi\oplus\Xi_{w_0.\phi})$.

References. This description of the cohomology of the boundary with complex coefficients for imaginary quadratic F has been extracted from the proof of Theorem 2 in [HaGL2] and [F] §1.5. □

2.10.2 Eisenstein operator

We now have for $\operatorname{Re}(z)\gg 0$ an operator

$$\operatorname{Eis}: V_{\phi|\alpha|^{z/2}}^{K_f}\to C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$$

given by the formula

$$\Psi\mapsto \operatorname{Eis}(\Psi)(g)=\sum_{\gamma\in B(\mathbf{Q})\backslash G(\mathbf{Q})}\Psi(\gamma g).$$

This can be meromorphically continued to all $z\in\mathbf{C}$. The image lies, in fact, in the space of automorphic forms $\mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$, a subspace of $C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$ of functions of moderate growth (see [Schw] I §4.2). (For $m=n$, see [U95] (3.1.1) for the relation of these scalar-valued automorphic forms to the ones defined in Section 2.7.) Via the map on cocycles the operator induces a map on $(\mathfrak{g}_\infty, K_\infty)$-cohomology

$$H^*(\mathfrak{g}_\infty, K_\infty, V_{\phi,\mathbf{C}}^{K_f}\otimes M_{\mathbf{C}}^{m,n})\xrightarrow{\operatorname{Eis}} H^*(\mathfrak{g}_\infty, K_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)\otimes M_{\mathbf{C}}^{m,n}).$$

The map factors through

$$\Xi_\phi : H^*(\mathfrak{g}_\infty, K_\infty, V_{\phi,\mathbf{C}}^{K_f} \otimes M_{\mathbf{C}}^{m,n}) \to H^*(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f) \otimes M_{\mathbf{C}}^{m,n})$$

and the direct sum of the images of Eis over all ϕ in $S_1(m,n,0,0)$ and $\overline{S}_1(m,n,0,0)$ is called the Eisenstein cohomology. It is shown in [HaGL2] Theorem 2, that the restriction of this part (for sufficiently small K_f) already exhausts the image of the restriction map

$$\text{res} : H^*(S_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}}) \to H^*(\partial\tilde{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}}).$$

This shows the existence of a section. Note, however, that it is in general not given by Eis, as the following discussion will show.

For a cohomology class in $H^i(S_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$ represented by a relative Lie algebra i-cocycle $\omega \in \text{Hom}_{K_\infty}(\Lambda^i(\mathfrak{g}_\infty/\mathfrak{k}_\infty), C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f) \otimes M_{\mathbf{C}}^{m,n})$, the restriction map is given by the class of the constant term

$$\text{res}(\omega) \in \text{Hom}_{K_\infty}(\Lambda^i(\mathfrak{g}_\infty/\mathfrak{k}_\infty), C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f) \otimes M_{\mathbf{C}}^{m,n})$$

with

$$\text{res}(\omega)(g) = \int_{U(\mathbf{Q})\backslash U(\mathbf{A})} \omega(ug)\, du,$$

where du is a Haar measure such that the volume of $U(\mathbf{Q})\backslash U(\mathbf{A})$ is equal to 1 (see [Ha79] Proposition 1.6.1, [Z] Proposition 2.2.3).

For K_f sufficiently small and $\phi = (\mu_1, \mu_2)$, one shows for $\Psi \in V_\phi^{K_f}$ that on the level of functions

$$\text{res}(\text{Eis}(\Psi)) = \Psi + \star \frac{L(-1, \mu_1/\mu_2)}{L(0, \mu_1/\mu_2)} T_\phi \Psi \in V_\phi^{K_f} \oplus V_{w_0 \cdot \phi}^{K_f},$$

for an intertwining operator $T_\phi : V_\phi^{K_f} \to V_{w_0\cdot\phi}^{K_f}$ and some non-zero factor $\star$. We will calculate this explicitly for some specific cohomology classes in the next chapter. Note that this differs significantly from the situation for $\text{GL}_{2,\mathbf{Q}}$ in [S02a], where $\text{res}(\text{Eis}(\Psi)) = \Psi$ (due to a pole of the L-function in the denominator).

CHAPTER III

Eisenstein cohomology

We write down an explicit Eisenstein cohomology class and calculate its constant term and Hecke eigenvalues. We also investigate the integrality of the constant term by translating to group cohomology.

3.1 Some double coset decompositions

Given

$$\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$$

with an infinity type contributing to the boundary cohomology $H^1(\partial \overline{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$ we will consider (essentially) $K_f = K^1(\mathfrak{N}_1\mathfrak{N}_2)$, where $\mathfrak{N}_i$ is the conductor of μ_i.

For the definition of functions in $V_{\phi_f}^{K_f}$ we will need the following lemma.

Lemma 3.1. *Let L be a non-archimedean local field, let $\mathcal{O}_L$ be its ring of integers, and let $\mathfrak{P}$ be its unique maximal ideal. Let π be a uniformizer in L. Then for any $s \geq 0$*

$$\mathrm{GL}_2(L) = \coprod_{i=0}^{s} B(L)\gamma_i K^1(\mathfrak{P}^s),$$

where

$$K^1(\mathfrak{P}^s) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_L), a-1, c \equiv 0 \mod \mathfrak{P}^s \right\}$$

and for $i = 0, \ldots, s-1$, $\gamma_i = \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix}$ and $\gamma_s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Note also that $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ *is in the same double coset as* $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

Proof. We first claim that

$$\mathrm{GL}_2(\mathcal{O}_L) = \coprod_{i=0}^{s} B(\mathcal{O}_L)\gamma_i K^1(\mathfrak{P}^s).$$

We prove this using the argument in Lemma 4.4 of [PR]. If $k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_L)$, and if $c \in \mathfrak{P}^i \backslash \mathfrak{P}^{i+1}$ for $1 \leq i \leq s-1$, then noting that $d \in \mathcal{O}_L^*$ and $c\pi^{-i} \in \mathcal{O}_L^*$, we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - bcd^{-1} & bcd^{-1}\pi^{-i} \\ 0 & c\pi^{-i} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi^i & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & c^{-1}d\pi^i \end{pmatrix},$$

which shows that $k \in B(\mathcal{O}_L)\gamma_i K^1(\mathfrak{P}^s)$.

If $c \in \mathcal{O}_L^*$, multiplying k on the left by $\begin{pmatrix} c^{-1} & 0 \\ 0 & c^{-1} \end{pmatrix}$, we may assume that $c = 1$. In this case, we have

$$\begin{pmatrix} a & b \\ 1 & d \end{pmatrix} = \begin{pmatrix} ad - b & a - (1+\pi^s)(ad-b) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\pi^s & d(1+\pi^s) - 1 \\ -\pi^s & 1 - d\pi^s \end{pmatrix}.$$

Lastly, one can check that if $c \in \mathfrak{P}^s$, then $k = \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1+\pi^s a' & b' \\ c & d \end{pmatrix}$ for some $x \in \mathcal{O}_L^*, y, a', b' \in \mathcal{O}_L$.

Using the Iwasawa decomposition $\mathrm{GL}_2(L) = B(L)\mathrm{GL}_2(\mathcal{O}_L)$ we get our result after checking that the double cosets are disjoint. □

The following Lemma will be needed to prove that our choice for $\Psi \in V_{\phi_f}^{K^1(\mathfrak{N})}$ is well-defined.

Lemma 3.2. *For* $0 \leq i \leq s$, $\gamma_i K^1(\mathfrak{P}^s)\gamma_i^{-1} \cap B(L)$ *is the subgroup*

$$\left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in \mathrm{GL}_2(\mathcal{O}_L) : d - 1, b - (1-a)/\pi^i \equiv 0 \mod \mathfrak{P}^{s-i}, a \equiv 1 \mod \mathfrak{P}^i \right\}.$$

Proof. See Lemma 4.5 in [PR]. □

For our Hecke operator calculations we will also need

Lemma 3.3. *Let L be a non-archimedean local field, $\mathcal{O}_L$ its ring of integers and $\mathfrak{P}$ its unique maximal ideal. Denote by π a uniformizer in L. Then for $K^0 = \mathrm{GL}_2(\mathcal{O}_L)$ and $K^1(\mathfrak{P}^s)$ as before, we have*

$$\begin{cases} K^0 \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K^0 = K^0 \begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix} \bigsqcup_{a \in \mathcal{O}_L/\mathfrak{P}} K^0 \begin{pmatrix} 1 & a \\ 0 & \pi \end{pmatrix} \\ K^1(\mathfrak{P}^s) \begin{pmatrix} 1 & 0 \\ 0 & \pi \end{pmatrix} K^1(\mathfrak{P}^s) = \bigsqcup_{a \in \mathcal{O}_L/\mathfrak{P}} K^1(\mathfrak{P}^s) \begin{pmatrix} 1 & a \\ 0 & \pi \end{pmatrix} & \text{if } s > 0. \end{cases}$$

Proof. See [Miy2] Lemma 2. □

3.2 An explicit boundary cohomology class

Let $\mu_1, \mu_2 : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ be two characters such that $\phi = (\mu_1, \mu_2) : T(\mathbf{Q}) \backslash T(\mathbf{A}) \to \mathbf{C}^*$ has an infinity type contributing to the boundary cohomology $H^1(\partial \tilde{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$ for a suitable K_f. In this section we want to write down explicit representatives for elements in the ϕ-part of the boundary cohomology. By Proposition 2.12 cocycles with non-trivial cohomology class can be described by certain elements in $\mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, V_{\phi,\mathbf{C}}^{K_f} \otimes M_{\mathbf{C}}^{m,n})$ if $V_{\phi_f}^{K_f} \neq 0$. We will try to choose K_f as large as possible (to keep down the number of connected components of S_{K_f}) but such that the boundary cohomology is still non-zero. Since we also want our cohomology class to be an eigenform for the Hecke operators we are led to choose newvectors at almost all places. At places v where both μ_i are ramified but $(\mu_1/\mu_2)_v$ is unramified, we take certain spherical vectors (used already by Harder in [Ha82] §4.6). Requiring K_∞-invariance will lead us to write down an element in the ϕ-part of the boundary cohomology.

Observe that with $\mathfrak{p} \cong \mathfrak{g}_\infty/\mathfrak{k}_\infty$ we have $\mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty, K_\infty, V_{\phi,\mathbf{C}}^{K_f} \otimes M_{\mathbf{C}}^{m,n}) \cong (\check{\mathfrak{p}}_{\mathbf{C}} \otimes_{\mathbf{C}} M_{\mathbf{C}}^{m,n} \otimes_{\mathbf{C}} V_{\phi,\mathbf{C}}^{K_f})^{K_\infty}$. (From now on the subscript $\mathbf{C}$ on V_ϕ will be suppressed, but all the induced modules in this section are understood to be $\mathbf{C}$-vector spaces.) Following

[HaGL2] p. 80 and [Ko] p. 101 we therefore define

$$\omega_z(\cdot,\phi,\Psi) : G(\mathbf{A}) \to \check{\mathfrak{p}}_{\mathbf{C}} \otimes_{\mathbf{C}} M_{\mathbf{C}}^{m,n}$$

for $z \in \mathbf{C}$ and $\Psi \in V^{K_f}_{\phi_f|\alpha|_f^{z/2}}$ as

$$\begin{aligned}
(3.1)\qquad & \omega_z(g,\phi,\Psi) := \omega(b_\infty k_\infty \cdot g_f, \phi|\alpha|^{z/2}, \Psi) = \\
= \quad & (\phi_\infty \cdot |\alpha|_\infty^{z/2})(b_\infty)\cdot \Psi(g_f) \begin{cases} k_\infty^{-1}.\,(\check{S}_+ \otimes Y^m \overline{X}^n) & \text{if } \phi \in S_1, \\ k_\infty^{-1}.\,((-\check{S}_-) \otimes X^m \overline{Y}^n) & \text{if } \phi \in \overline{S}_1 \end{cases}.
\end{aligned}$$

Here $S_1 = S_1(m,n,0,0)$ and $\overline{S}_1 = \overline{S}_1(m,n,0,0)$ are the two different infinity types contributing to the boundary cohomology (cf. Section 2.10.1),

$$S_\pm = 1/2\left(\pm\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\otimes_{\mathbf{R}} 1 - \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\otimes_{\mathbf{R}} i\right) \in \mathfrak{p}_{\mathbf{C}},$$

and the '$\vee$' denotes the dual vectors with respect to the killing form (see Section 2.4). The above elements of $\check{\mathfrak{p}}_{\mathbf{C}} \otimes_{\mathbf{C}} M_{\mathbf{C}}^{m,n}$ are related to generators of $H^1(\mathfrak{u}_\infty, M_{\mathbf{C}}^{m,n})$ as used in Proposition 2.12 under the isomorphism

$$\mathfrak{u}_\infty \oplus \mathbf{C}H \cong \mathfrak{b}_\infty/(\mathfrak{b}_\infty \cap \mathfrak{k}_\infty) \cong \mathfrak{g}_\infty/\mathfrak{k}_\infty \cong \mathfrak{p}_\infty$$

induced by the embedding $\mathfrak{b}_\infty \hookrightarrow \mathfrak{g}_\infty$ (cf. [Ko] §1.3.6). For the definition of $V^{K_f}_{\phi_f|\alpha|_f^{z/2}}$ see (2.9). One checks that if $z = 0$ then ω_0 is a relative Lie algebra 1-cocycle (see [Ha79] Lemma 1.5.2). We write $[\omega_0(\phi,\Psi)]$ both for the corresponding cohomology class in $H^1(\mathfrak{g}_\infty, \mathfrak{k}_\infty, V_\phi \otimes M_{\mathbf{C}}^{m,n})$ as well as its image under Ξ_ϕ in $H^1(\partial\tilde{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$.

Given $\eta = (\eta_1,\eta_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ we now want to fix a choice of K_f and $\Psi \in V_{\eta_f}^{K_f}$ for which the constant term will have a particularly nice form. The function will be denoted by Ψ_{η_f}. As K_f we will take $K^1(\mathfrak{M}_1\mathfrak{M}_2)$ (at least away from finitely many places), where $\mathfrak{M}_i$ is the conductor of η_i. We will drop the subscript η_f if it is clear what is meant from the context.

We define Ψ_{η_f} as a product of local factors $\prod_v \Psi_{\eta,v}$. We will also use the notation $V_{\eta_v} = \{\Psi_v : G(F_v) \to \mathbf{C} \mid \Psi(b_v g_v) = \eta_v(b_v)\Psi(g_v)\}$, and denote the elements $\Psi_{\eta,v} \in V_{\eta_v}$ sometimes by Ψ_{η_v}.

(a) If $(\eta_1/\eta_2)_v$ is ramified we make use of the adelic interpretation of Atkin-Lehner theory (see [Cas] §1). For $\mathfrak{P}_v^s \parallel \mathfrak{M}_1\mathfrak{M}_2$ Casselman shows that $V_{\eta_v}^{K^1(\mathfrak{P}_v^s)}$ is 1-dimensional. Let $\Psi_v : \mathrm{GL}_2(F_v) \to \mathbf{C}$ be the newvector spanning $V_{\eta_v}^{K^1(\mathfrak{P}_v^s)}$:

$$\Psi_v(g) = \begin{cases} \eta_{1,v}(a)\eta_{2,v}(d) & \text{if } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} k, k \in K^1(\mathfrak{P}_v^s) \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $\mathfrak{P}_v^r \parallel \mathfrak{M}_1$ and $\mathfrak{P}_v^s \parallel \mathfrak{M}_1\mathfrak{M}_2$.

(b) If $\eta_{1,v}/\eta_{2,v}$ is unramified (but possibly both characters are ramified) we make a different choice. For an ideal $\mathfrak{N}_v \subset \mathcal{O}_v$ put

$$U^1(\mathfrak{N}_v) = \{k \in \mathrm{GL}_2(\mathcal{O}_v) : \det(k) \equiv 1 \mod \mathfrak{N}_v\}.$$

Then there is a distinguished function spanning $V_{\eta_v}^{U^1(\mathfrak{M}_{1,v})}$, the spherical function:

$$\Psi_v^0(g) = \eta_{1,v}(a)\eta_{2,v}(d)\eta_{1,v}(\det(k)) \text{ for } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, k \in \mathrm{GL}_2(\mathcal{O}_v). \tag{3.3}$$

Note that if both η_i are unramified then the spherical function equals the newvector.

For future reference we record:

Definition 3.4. Denote by S the set of places where both η_i are ramified but η_1/η_2 is unramified. Then $\Psi_{\eta_f} := \prod_{v\in S} \Psi_v^0 \prod_{v\notin S} \Psi_v$ is in $V_{\eta_f}^{K_f^s}$ for

$$K_f^s := \prod_{v\in S} U^1(\mathfrak{M}_{1,v}) \prod_{v\notin S} K^1((\mathfrak{M}_1\mathfrak{M}_2)_v).$$

3.3 An Eisenstein cohomology class and its constant term

3.3.1 Definition of Eisenstein cohomology classes

In Section 2.10 we defined the Eisenstein operator $\mathrm{Eis} : \mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, V_{\phi|\alpha|^{z/2}}^{K_f} \otimes M_{\mathbf{C}}^{m,n}) \to \mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega^{-1}) \otimes M_{\mathbf{C}}^{m,n})$. We introduce the notation $\mathrm{Eis}(\phi|\alpha|^{z/2}, \Psi) := \mathrm{Eis}(\omega_z(\phi, \Psi))$ for $\Psi \in V_{\phi|\alpha|^{z/2}}^{K_f}$. Harder shows in [HaGL2]

Theorem 2 that for $z = 0$ we get a holomorphic closed form. For $g \in G(\mathbf{A})$ and $A \in \mathfrak{g}_\infty/\mathfrak{k}_\infty$ we use the notation $\mathrm{Eis}(g, \phi, \Psi)(A)$ for the Lie algebra 1-cocycle in $\mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega^{-1}) \otimes M_{\mathbf{C}}^{m,n})$. The corresponding de Rham 1-form in $\Omega^1(S_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$ is denoted by $\mathrm{Eis}(x, \phi, \Psi)(\theta_x)$ for $x \in S_{K_f}$ and $\theta_x \in T_x S_{K_f}$. We write $[\mathrm{Eis}(\phi, \Psi)]$ for the associated cohomology class in $H^1(S_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$. We will later drop the ϕ in the argument if it is clear from the context.

3.3.2 Constant term

As indicated in Section 2.10, the restriction to the boundary of a cohomology class $[\omega] \in H^1(S_{K_f}, \widetilde{M}_{\mathbf{C}}^{m,n})$, ω a Lie algebra cocycle, is the class

$$[\mathrm{res}(\omega)] \in H^1(B(\mathbf{Q})\backslash G(\mathbf{Q})/K_f K_\infty, \widetilde{M}_{\mathbf{C}}^{m,n})$$

with

$$\mathrm{res}(\omega)(g) = \int_{U(\mathbf{Q})\backslash U(\mathbf{A})} \omega(ug)du.$$

To calculate the constant term of the Eisenstein series we build on the calculations in [HaGL2] Theorem 2, [Ha82] Theorem 2 and [Ha79] Theorem 2.1. We describe in detail the calculations at the ramified places. First we introduce an intertwining operator $T_{\eta_f} : V_{\eta_f} \to V_{w_0.\eta_f}$. (Recall that $w_0.(\eta_1, \eta_2) = (\eta_2 \cdot |\,|, \eta_1 \cdot |\,|^{-1})$ and note that $w_0.(\phi|\alpha|^{z/2}) = (w_0.\phi)|\alpha|^{-z/2}$.) This operator is defined as a product of local intertwining operators T_{η_v}. If η_1/η_2 is unramified, we require that T_{η_v} maps the spherical function $\Psi^0_{\eta_v}$ in $V_{\eta_v}^{K_f}$ to $\Psi^0_{w_0.\eta_v}$. At places where η_1/η_2 is ramified we put $T_{\eta_v}(\Psi_v)(g_v) = \int_{U_0(F_v)} \Psi_v(w_0 \cdot u_v g_v)du_v$ (cf. [Ha82] pp. 114/5, [HaGL2] pp.76, 81).

Proposition 3.5. *The constant term for our specific choice of* $\Psi = \Psi_{\phi_f|\alpha|_f^{z/2}}$ *is given by*

$$\mathrm{res}(\mathrm{Eis}(g, \phi \cdot |\alpha|^{z/2}, \Psi)) = \omega_z(g, \phi, \Psi) + c(\phi, z)\omega_{-z}(g, w_0.\phi, T_{\phi_f|\alpha|_f^{z/2}}(\Psi))$$

$$= \omega_z(g, \phi, \Psi) + d(\phi, z)\omega_{-z}(g, w_0.\phi, \Psi_{w_0.(\phi_f|\alpha|_f^{z/2})}),$$

where

$$c(\phi,z)=(d_F)^{-1/2}\frac{2\pi}{z+m+1}(-1)^{n+1}\cdot\frac{L(\mu_1/\mu_2,z-1)}{L(\mu_1/\mu_2,z)}$$

and

$$d(\phi,z)=c(\phi,z)\cdot\prod_{(\mu_1/\mu_2)_v\ \textit{ramified}} d_v(\phi)$$

for $d_v(\phi):=T_{\phi_v|\alpha|_v^{z/2}}(\Psi_v)(\begin{pmatrix}1&0\\ \pi_v^{s-r}&1\end{pmatrix})$ *if* $\mathfrak{P}_v^r\parallel\mathfrak{N}_1$ *and* $\mathfrak{P}_v^s\parallel\mathfrak{N}_1\mathfrak{N}_2$*, where* $\mathfrak{N}_i$ *is the conductor of* μ_i*. If only one of* $\{\mu_{1,v},\mu_{2,v}\}$ *is ramified then* $d_v(\phi)=\frac{\mu_{2,v}(-1)}{\mathrm{Nm}(\mathfrak{P}_v^r)}$.

Proof. Using the Bruhat decomposition we have

$$\mathrm{res}(\mathrm{Eis}(g,\phi\cdot|\alpha|^{z/2},\Psi))=\int_{U(\mathbf{Q})\backslash U(\mathbf{A})}\mathrm{Eis}(ug,\phi\cdot|\alpha|^{z/2},\Psi)du$$

$$=\omega_z(g,\phi,\Psi)+\int_{U(\mathbf{A})}\omega_z(w_0ug,\phi,\Psi)du$$

and we obtain the first part of $c(\phi)$ from the calculation at the infinite place, which is done in [Ha79] pp.71-72 and [HaGL2] pp. 71-73.

At the finite places where μ_1/μ_2 is unramified, it is a standard calculation (cf. [B92] pp. 478, 497) that the integral $\int_{U_0(F_v)}\Psi^0_{\phi_v|\alpha|_v^{z/2}}(w_0u_vg_v)du_v$ gives a multiple of the corresponding spherical function $\Psi^0_{w_0\cdot\phi_v|\alpha|_v^{-z/2}}$ (which equals $T_{\phi_v|\alpha|_v^{z/2}}(\Psi^0_{\phi_v|\alpha|_v^{z/2}})$ by definition), the factor being given by

$$\int_{U_0(F_v)}\Psi_v^0(w_0u_v)du_v=\frac{L_v(\mu_1/\mu_2,z-1)}{L_v(\mu_1/\mu_2,z)}.$$

We now give the calculation for $\mu_{1,v}/\mu_{2,v}$ ramified. It is easy to check that

$$T_{\phi_v|\alpha|_v^{z/2}}:V^{K^1(\mathfrak{P}_v^s)}_{\phi_v|\alpha|_v^{z/2}}\to V^{K^1(\mathfrak{P}_v^s)}_{w_0\cdot(\phi_v|\alpha|_v^{z/2})},$$

so $T_{\phi_v|\alpha|_v^{z/2}}(\Psi_v)(g)$ must be a multiple of

$$\Psi_{w_0\cdot(\phi_v|\alpha|_v^{z/2})}(g)=\begin{cases}\mu_{2,v}(a)\mu_{1,v}(d)|\frac{a}{d}|^{1-z/2} & \text{if } g=\begin{pmatrix}a&b\\0&d\end{pmatrix}\begin{pmatrix}1&0\\ \pi_v^{s-r}&1\end{pmatrix}k, k\in K^1(\mathfrak{P}_v^s)\\ 0 & \text{otherwise,}\end{cases}$$

the newvector spanning $V^{K^1(\mathfrak{P}_v^s)}_{w_0\cdot(\phi_v|\alpha|_v^{z/2})}$. The multiple is given by

$$d_v(\phi) = T_{\phi_v|\alpha|_v^{z/2}}(\Psi_v)(\begin{pmatrix} 1 & 0 \\ \pi_v^{s-r} & 1 \end{pmatrix}).$$

Next note that

$$\begin{aligned} T_{\phi_v|\alpha|_v^{z/2}}(\Psi_v)(g_v) &= \int_{U_0(F_v)} \Psi_v(w_0 u_v g_v) du_v \\ &= \int_{U_0(\mathcal{O}_v)} \Psi_v(w_0 u_v g_v) du_v + \sum_{n=1}^{\infty} \int_{\pi_v^{-n}\mathcal{O}_v^*} \Psi_v(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g_v) dx. \end{aligned}$$

The important cases for us are:

(I) $r = 0$, i.e., $\mu_{1,v}$ is unramified, but $s > 0$

(II) $s - r = 0$, i.e., $\mu_{2,v}$ is unramified, but $r > 0$

The situation for other values of r and s is messy; in certain cases $d_v(\phi)$ can be zero. We will always be able to put ourselves in the situation of Case I or II.

In Case I we have that $[\begin{pmatrix} 1 & 0 \\ \pi_v^{s-r} & 1 \end{pmatrix}] = [\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}]$ ('[ ]' indicating the double coset in $B(F_v)\backslash G(F_v)/K^1(\mathfrak{P}_v^s)$), so we can determine $d_v(\phi)$ by evaluating at the identity matrix. The terms in the infinite sum all turn out to be zero since for $n \geq 1$ and $x \in \mathcal{O}_v^*$

$$w_0 \cdot \begin{pmatrix} 1 & \pi_v^{-n}x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\pi_v^n x^{-1} & 1 \\ 0 & -\pi_v^{-n}x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_v^n x^{-1} & 1 \end{pmatrix}$$

does not lie in the same double coset as $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and so gets mapped to zero by Ψ. The constant $d_v(\phi)$ is therefore given by

$$\int_{\mathcal{O}_v} \Psi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}) dx = \int_{\mathcal{O}_v} \Psi(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) dx =$$
$$\mathrm{vol}(\mathcal{O}_v)\mu_{1,v}(-1)\mu_{2,v}(-1) = \mu_{2,v}(-1),$$

where we have used that

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 1+\pi^s \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\pi^s & -1 \\ -\pi^s & 1 \end{pmatrix}.$$

In Case II we can evaluate at $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. For ease of calculation we calculate $T_{\phi_v|\alpha|_v^{z/2}}(\Psi_v)(w_0)$ which gives us $d_v(\phi)$ up to a factor of $\mu_{1,v}\mu_{2,v}(-1)$, which equals $\mu_{1,v}(-1)$ by assumption. Again the infinite sum does not contribute anything since

$$w_0 \begin{pmatrix} 1 & \pi_v^{-n}x \\ 0 & 1 \end{pmatrix} w_0 = \begin{pmatrix} -\pi_v^n x^{-1} & 1 \\ 0 & -\pi_v^{-n}x \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & \pi_v^n x^{-1} \end{pmatrix}$$

lies in the double coset $[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}]$, which is different from $[\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}]$ in this case. We are left with

$$\begin{aligned} \int_{\mathcal{O}_v} \Psi_v(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} w_0)dx &= \int_{\mathcal{O}_v} \Psi_v(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -x & 1 \end{pmatrix})dx = \\ &= \int_{\mathfrak{P}_v^r} \mu_{1,v}(-1)\mu_{2,v}(-1)dx = \frac{\mu_{1,v}(-1)}{\mathrm{Nm}(\mathfrak{P}_v^r)}. \end{aligned}$$

Both cases can therefore be summarized by

$$T_{\phi_v|\alpha|_v^{z/2}}(\Psi_{\phi_v|\alpha|_v^{z/2}}) = \frac{\mu_{2,v}(-1)}{\mathrm{Nm}(\mathfrak{P}_v^r)} \cdot \Psi_{(w_0.\phi_v)|\alpha|_v^{-z/2}}.$$

□

3.3.3 Restriction to particular boundary components

Recall from (2.5) that the boundary of the Borel-Serre compactification of the adelic symmetric space is given by

$$\partial \overline{S}_{K_f} = \coprod_{[\det(\xi)] \in H_K} \coprod_{[\eta] \in \mathbf{P}^1(F)/\Gamma_\xi} \Gamma_{\xi, B^\eta} \backslash e(B^\eta),$$

where $H_K = F^* \backslash \mathbf{A}_F^* / \det(K_f)\mathbf{C}^*$, $\Gamma_\xi = G(\mathbf{Q}) \cap \xi K_f \xi^{-1}$ for $\xi \in G(\mathbf{A}_f)$, $\Gamma_{\xi,P} = \Gamma_\xi \cap P(\mathbf{Q})$ for parabolic subgroups $P/\mathbf{Q}$, and $B^\eta(\mathbf{Q}) = \eta^{-1}B(\mathbf{Q})\eta$ for $\eta \in G(\mathbf{Q})$. This is homotopy equivalent to

$$\partial \tilde{S}_{K_f} = B(\mathbf{Q}) \backslash G(\mathbf{Q}) / K_f K_\infty \cong \coprod_{[\det(\xi)] \in H_K} \coprod_{[\eta] \in \mathbf{P}^1(F)/\Gamma_\xi} \Gamma_{\xi, B^\eta} \backslash \mathbf{H}_3,$$

where the boundary component $\Gamma_{\xi,B^\eta}\backslash\mathbf{H}_3$ gets embedded in $\partial\tilde{S}_{K_f}$ via $g_\infty \mapsto j_{\eta,\xi}(g_\infty) := \eta(g_\infty,\xi)$.

We will be interested in the restriction of cohomology classes to the boundary components $\Gamma_{\xi,P}\backslash e(P) \sim \Gamma_{\xi,P}\backslash\mathbf{H}_3$. So in the next lemma we clarify the relation between the various descriptions of the boundary restrictions of a class in $H^1(\overline{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$.

Lemma 3.6 (Definition/Lemma). (a) *For* $[\omega] \in H^1(\partial\tilde{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$*,* ω *a Lie algebra cocycle, the restriction to* $H^1(\Gamma_{\xi,P}\backslash\mathbf{H}_3, \widetilde{M_{\mathbf{C}}^{m,n}})$ *is given by the class of* $\omega_P^\xi := j_{\eta,\xi}^*(\omega)$ *if* $P = B^\eta$*.*

(b) *For* $[\omega] \in H^1(\overline{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$*,* ω *a Lie algebra cocycle, the restriction to*

$$H^1(\Gamma_{\xi,P}\backslash\mathbf{H}_3, \widetilde{M_{\mathbf{C}}^{m,n}})$$

is given by the class of

$$\mathrm{res}_P^\xi(\omega)(g) = \int_{U_P(\mathbf{Q})\backslash U_P(\mathbf{A})} \omega(ug\xi)du$$

for $g \in G_\infty$*, where* U_P *is the nilpotent part of the parabolic* P*. We have* $\mathrm{res}_P^\xi(\omega) = (\mathrm{res}(\omega))_P^\xi$*, where* $\mathrm{res}(\omega)$ *is the constant term defined in section 2.10.*

Remark 3.7. The "constant term of ω with respect to P" is given by

$$\int_{U_P(\mathbf{Q})\backslash U_P(\mathbf{A})} \omega(ug)du.$$

Since we will not be using this general notion, "constant term" will always refer to the one with respect to B, as defined in Section 2.10.

Proof. If one takes for granted the statements about the constant term recalled in Section 2.10 and the comments at the start of this section on the embedding of the boundary component, then the restriction to $\Gamma_{\xi,P}$ for $P = B^\eta$ is given by $\mathrm{res}(\omega)(j_{\eta,\xi}(g)) = \int_{U(\mathbf{Q})\backslash U(\mathbf{A})} \omega(u\eta g\xi)du$. The latter equals $\int_{U_P(\mathbf{Q})\backslash U_P(\mathbf{A})} \omega(ug\xi)du$ since ω is invariant under multiplication by η^{-1} on the left.

Alternatively, one can refer to Proposition 2.2.3 of [Z] which proves that the restriction of $[\omega]|_{j_\xi(\Gamma_\xi\backslash\mathbf{H}_3)}$ to the boundary component $\Gamma_{\xi,P}\backslash\mathbf{H}_3$ is given by the class of

$$\int_{\Gamma_{\xi,U_P}\backslash U_P(\mathbf{R})} \omega(u_\infty g\xi)du_\infty.$$

Strong approximation for $U_P(\mathbf{A})$ shows that this agrees with the expression given above. □

3.3.4 Translation to group cohomology

By Proposition 2.5 the de Rham cohomology group $H^1(\Gamma_{\xi,P}\backslash\mathbf{H}_3, \tilde{N})$ is naturally isomorphic to the cohomology group $H^1(\Gamma_{\xi,P}, N)$ for any $\mathbf{C}[\Gamma_{\xi,P}]$-module N. After a choice of basepoint $x_0 \in \mathbf{H}_3$ this isomorphism is given by mapping a closed 1-form $\tilde{\omega}$ with values in N to the following 1-cocyle:

$$\mathcal{G}_{x_0}(\tilde{\omega}) : \gamma \mapsto \int_{x_0}^{\gamma.x_0} \tilde{\omega}.$$

For later considerations it will be convenient to translate the restrictions ω_P^ξ of adelic boundary cohomology classes ω to group cohomology for the arithmetic subgroups $\Gamma_{\xi,P}$, i.e. to make the isomorphism

$$H^1(\partial\tilde{S}_{K_f}, \widetilde{M_{\mathbf{C}}}) \cong \bigoplus_{[\det(\xi)]\in H_K} \bigoplus_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\xi} H^1(\Gamma_{\xi,B^\eta}, M_\xi \otimes \mathbf{C}) \tag{3.4}$$

(cf. Section 2.9.2) explicit for M an $\mathcal{O}[\frac{1}{6}, G(\mathbf{Q})]$-module. Note that $M_\xi \otimes C = M_{\mathbf{C}}$.

To combine this with the results of the previous section on the restriction of Lie algebra cohomology classes to a boundary component, recall from Section 2.9.3 how to translate a Lie algebra cocycle $\omega \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, C^\infty(\Gamma\backslash\mathrm{GL}_2(\mathbf{C}))(\omega_\infty^{-1}) \otimes M_{\mathbf{C}})$ to a closed 1-form $\tilde{\omega}$: Reversing the isomorphism given there, for $x_\infty = g_\infty K_\infty \in \mathbf{H}_3$ and $T \in T_{x_\infty}\mathbf{H}_3$ let

$$\tilde{\omega}(x_\infty)(T) := g_\infty.\omega(g_\infty)(\mathrm{D}_{\mathrm{L}_{g_\infty}^{-1}} T).$$

For a particular choice of basepoint we then get a fairly nice expression for the image of a boundary cohomology class in $H^1(\partial\tilde{S}_{K_f}, \widetilde{M_{\mathbf{C}}^{m,n}})$ (represented by a relative Lie algebra 1-cocycle ω) in the group cohomology of $\Gamma_{\xi,P}$:

Lemma 3.8. *For $P = B^\eta$ let $x_0 = \eta_\infty^{-1} K_\infty$. Then $\mathcal{G}_{x_0}\tilde{\omega}_P^\xi$ is given on U^η by*

$$\eta_\infty^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta_\infty \mapsto \int_0^1 (\begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}).\omega(\begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}, \eta_f\xi)(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} dt.$$

Here we denote by η_f and η_∞ the images of $\eta \in G(\mathbf{Q})$ in $G(\mathbf{A}_f)$ and G_∞ respectively.

Proof. By definition,

$$\mathcal{G}_{\eta_\infty^{-1}K_\infty}(\tilde{\omega}_P^\xi)(\eta_\infty^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta_\infty) = \int_{\eta_\infty^{-1}K_\infty}^{\eta_\infty^{-1}\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)K_\infty} \tilde{\omega}_P^\xi.$$

To calculate the path integral we apply the following lemma, adapted from [Wes]:

Lemma 3.9 ([Wes] Lemma 5.1). *Given $h : \mathbf{R} \to G_\infty$ a differentiable homomorphism and $g \in G_\infty$, define $c : \mathbf{R} \to \mathbf{H}_3$ by $c(t) := h(t) \cdot g \cdot x^0$. For $a_0, a_1 \in \mathbf{R}$ let $y_i := c(a_i)$ and denote $\dot{h} := (Dh)_0 T_0 \in \mathfrak{g}_\infty$. Then one has the following equality:*

$$\int_{y_0}^{y_1} \tilde{\omega} = \int_{a_0}^{a_1} (h(t)g).\omega(h(t)g, g_f)(g^{-1}\dot{h}g)dt.$$

We take $y_0 = \eta_\infty^{-1}$, $g = \eta_\infty^{-1}$, $h(t) = \eta_\infty^{-1} \begin{pmatrix} 1 & xt \\ 0 & 1 \end{pmatrix} \eta_\infty \in G_\infty$, $a_0 = 0$, $a_1 = 1$, and obtain:

$$\mathcal{G}_{\eta_\infty^{-1}K_\infty}(\tilde{\omega}_P^\xi)(\eta_\infty^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \eta_\infty) = \int_0^1 (\eta_\infty^{-1} \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix}).\omega_P^\xi(\eta_\infty^{-1} \begin{pmatrix} 1 & tx \\ 0 & 1 \end{pmatrix})(\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix})dt.$$

With $\omega_P^\xi(g_\infty) = \omega(\eta g_\infty \xi)$ we get the expression given in the lemma.

□

We record for later:

Lemma 3.10. *For* $\phi = (\phi_1, \phi_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$, $\Psi \in V_{\phi_f}^{K_f}$, *and* $\omega = \omega_0(\phi, \Psi)$ *(see (3.1)) we get that the image of* $[\omega]$ *in* $H^1(\Gamma_{\xi,B^\eta}, M_{\mathbf{C}}^{m,n}))$ *is represented by the 1-cocycle*

$$\mathcal{G}_{\eta_\infty^{-1}K_\infty}(\tilde{\omega}^\xi_{B^\eta})(\eta_\infty^{-1}\begin{pmatrix} a & x \\ 0 & d\end{pmatrix}\eta_\infty) = \Psi(\eta_f\xi)\begin{cases} x\int_0^1 \begin{pmatrix} 1 & tx \\ 0 & 1\end{pmatrix}.Y^m\overline{X}^n dt & \text{if } \phi \in S_1, \\ \overline{x}\int_0^1 \begin{pmatrix} 1 & tx \\ 0 & 1\end{pmatrix}.X^m\overline{Y}^n dt & \text{if } \phi \in \overline{S}_1. \end{cases}$$

Proof. Recall the definition of S_+ and S_- from Section 2.4. One checks that for $x \in \mathbf{C}$ one has

$$xS_+ - \overline{x}S_- = \begin{pmatrix} 0 & x/2 \\ \overline{x}/2 & 0\end{pmatrix} \equiv \begin{pmatrix} 0 & x \\ 0 & 0\end{pmatrix} \mod \mathfrak{k}_\infty.$$

In a similar manner as the preceding lemma one can show that $\mathcal{G}_{x_0}\tilde{\omega}^\xi_P$ is always zero on $\eta_\infty^{-1}\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\eta_\infty$. This follows from $\dot{h}$ being a multiple of H in this case, along which ω_0 vanishes.

□

3.4 Hecke eigenvalues of Eisenstein cohomology class

Next we want to calculate the effect of the Hecke operators on our Eisenstein class and its image under the restriction map. We recall briefly the definition of the Hecke operators from Section 2.9:

For a place v of F, a non-negative integer s and $g \in \mathrm{GL}_2(F_v)$ we define an action of the double coset $K^1(\mathfrak{P}_v^s)gK^1(\mathfrak{P}_v^s)$ on $f \in C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K^1(\mathfrak{P}_v^s))$ by

$$([K^1(\mathfrak{P}_v^s)gK^1(\mathfrak{P}_v^s)]f)(h) = \sum_i f(hx_i^{-1}),$$

where $K^1(\mathfrak{P}_v^s)gK^1(\mathfrak{P}_v^s) = \bigsqcup_i K^1(\mathfrak{P}_v^s)x_i$.

We are especially interested in the action of

$$T_{v,s} = T(\mathfrak{P}_v) = K^1(\mathfrak{P}_v^s) \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} K^1(\mathfrak{P}_v^s).$$

By the definition of the Eisenstein cohomology class $\mathrm{Eis}(\phi, \Psi)$, it suffices to check the effect of the Hecke operator $T_{v,s}$ on $\Psi_v \in V_{\phi_v|\alpha|_v^{z/2}}^{K^1(\mathfrak{P}_v^s)}$: Since our $\Psi_v \in V_{\phi_v|\alpha|_v^{z/2}}^{K^1(\mathfrak{P}_v^s)}$ are newvectors we get that

$$T_{v,s}(\Psi_v) = a_v(\phi|\alpha|^{z/2})\Psi_v \text{ for } a_v(\phi|\alpha|^{z/2}) \in \mathbf{C}.$$

For our application in Chapter VII we only need to consider places where either both μ_i are unramified or only one of them is ramified.

We first consider the places where both μ_i are unramified. We can therefore obtain $a_v(\phi|\alpha|^{z/2})$ by evaluating at the identity. We get by Lemma 3.3

$$a_v(\phi|\alpha|^{z/2}) = T_{v,0}(\Psi_v)(1) = \sum_i \Psi_v(x_i^{-1}) = \Psi_v(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}) + \sum_{a \in \mathcal{O}_v/\mathfrak{P}_v} \Psi_v(\begin{pmatrix} 1 & -\frac{a}{\pi_v} \\ 0 & \pi_v^{-1} \end{pmatrix}) =$$

$$= \mu_{1,v}(\pi_v^{-1}) \cdot |\pi_v^{-1}|^{z/2} + \mathrm{Nm}(\mathfrak{P}_v)\mu_{2,v}(\pi_v^{-1}) \cdot |\pi_v|^{z/2}$$

$$= \mathrm{Nm}(\mathfrak{P}_v)^{z/2}\mu_{1,v}(\pi_v^{-1}) + \mathrm{Nm}(\mathfrak{P}_v)^{1-z/2}\mu_{2,v}(\pi_v^{-1}).$$

At the places where μ_1 is unramified, i.e. $r = 0$, but μ_2 is ramified ($s > 0$) we get $a_v(\phi|\alpha|^{z/2})$ as value of $T_{v,s}(\Psi_v)(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix})$. However, like in the proof of Prop. 3.5 the computation is easier if we evaluate at w_0; obtaining the value at $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ by then multiplying by $\mu_{2,v}(-1)$. Using the second case of Lemma 3.3, we obtain

$$a_v(\phi|\alpha|^{z/2})\mu_{2,v}(-1) = T_{v,s}(\Psi_v)(w_0) = \sum_i \Psi_v(w_0 x_i^{-1}) = \sum_{a \in \mathcal{O}_v/\mathfrak{P}_v} \Psi_v(w_0 \begin{pmatrix} 1 & -\frac{a}{\pi_v} \\ 0 & \pi_v^{-1} \end{pmatrix}) =$$

$$= \Psi_v(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} w_0) + \sum_{a \in \mathcal{O}_v/\mathfrak{P}_v, a \neq 0} \Psi_v(\begin{pmatrix} a^{-1} & \pi_v^{-1} \\ 0 & a\pi_v^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\pi/a & 1 \end{pmatrix})$$

$$= \mu_{1,v}(\pi_v^{-1})\mu_{2,v}(-1)\mathrm{Nm}(\mathfrak{P}_v)^{z/2} + 0.$$

The case $r > 0, s-r = 0$ provides a useful check for our calculations. We can evaluate at the identity, and we get the eigenvalue $a_v(\phi|\alpha|^{z/2}) = \mathrm{Nm}(\mathfrak{P}_v)^{1-z/2}\mu_{2,v}(\pi^{-1})$.

We observe that all these eigenvalues are invariant if we replace $\phi|\alpha|^{z/2}$ with $w_0.(\phi|\alpha|^{z/2})$. This is explained by the fact shown in Prop. 3.5 that for all finite places there is some constant $d_v(\phi)$ so that

$$\Psi_{w_0.\eta_v} = d_v(\phi)\int_{U_0(F_v)} \Psi_{\eta_v}(w_0 \cdot u_v g_v) du_v.$$

Furthermore, this tells us that the image of the Eisenstein series under the restriction map has again the same eigenvalues for the action of the $T_{v,s}$'s on the boundary cohomology. To conclude we summarize our calculations:

Lemma 3.11. *For $\Psi_v \in V_{\phi_v|\alpha|_v^{z/2}}^{K^1(\mathfrak{P}_v^s)}$ we have $T_{v,s}(\Psi_v) = a_v(\phi|\alpha|^{z/2})\Psi_v$, where*

$$a_v(\phi|\alpha|^{z/2}) = \begin{cases} \mathrm{Nm}(\mathfrak{P}_v)^{z/2}\mu_{1,v}(\pi_v^{-1}) + \mathrm{Nm}(\mathfrak{P}_v)^{1-z/2}\mu_{2,v}(\pi_v^{-1}) & \text{if } r = s = 0, \\ \mathrm{Nm}(\mathfrak{P}_v)^{z/2}\mu_{1,v}(\pi_v^{-1}) & \text{if } r = 0, s > 0, \\ \mathrm{Nm}(\mathfrak{P}_v)^{1-z/2}\mu_{2,v}(\pi_v^{-1}) & \text{if } r = s > 0. \end{cases}$$

3.5 Examples and properties of algebraic Hecke characters

In this section we want to give a list of examples of Hecke characters and their properties that will be used in subsequent proofs.

Definition 3.12. We will call a Hecke character $\lambda : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ anticyclotomic if $\lambda^c = \overline{\lambda}$, where $\lambda^c(x) := \lambda(\overline{x})$.

Remark 3.13. 1. For finite order characters (i.e. $\lambda_\infty = 1$) this agrees with the usual definition (e.g. [Ti] Définition 3.4). In [dS] §II.6 a different notion ($\lambda = \lambda^*$) is used, one based on an involution of Hecke characters of type (A_0) preserving criticality: $\lambda \mapsto \lambda^*$, where $\lambda^*(x) = \lambda(\overline{x})^{-1}|x|_{\mathbf{A}_F}$. This arises in Katz's work [K76], in particular, in the p-adic functional equation. For infinity types z and $\overline{z}$ the two definitions agree (e.g. for the (inverse of) the Größencharakter arising from an elliptic curve with complex multiplication). This follows from $\lambda\overline{\lambda} = |\cdot|_{\mathbf{A}_F}^{m+n}$ if $\lambda_\infty = z^m\overline{z}^n$ (see below). Hida suggests in [Hi03] the definition $\lambda^c = \lambda^{-1}$, which, of course, agrees with our notion for unitary characters, relates, however, characters with different infinity types in the general case.

2. Note that our definition implies that the conductor of an anticyclotomic character λ is stable under complex conjugation. Furthermore, for infinity types with $\lambda_\infty(-1) = -1$ the conductor of λ must contain a non-split prime.

Lemma 3.14. *Let λ be a Hecke character of F with infinity type $\lambda_\infty = z^m\overline{z}^n$. Then $\lambda\overline{\lambda} = |\cdot|_{\mathbf{A}_F}^{m+n}$.*

Proof. Denote by (x) the fractional F-ideal generated by the (finite part) of an idele $x \in \mathbf{A}_F^*$ and by $\mathcal{O}_\lambda$ the ring of integers in the finite extension of F containing the values of the finite parts of λ and $\overline{\lambda}$.

We will show for each place v that $\lambda_v\overline{\lambda}_v(x_v) = |x_v|_v^{m+n}$. For the infinite place this is obvious. For finite places v we first note that λ has finite order on $\mathcal{O}_v^*$, so $\lambda\overline{\lambda}|_{\mathcal{O}_v^*}$ is trivial. Take now a uniformizer π_v. If h is the class number of F, we can write $(\pi_v)^h = (\alpha)$ for $\alpha \in \mathcal{O} \hookrightarrow \mathbf{A}_F^*$. One checks that $\lambda_v(\alpha)\mathcal{O}_\lambda = \alpha^{-m}\overline{\alpha}^{-n}\mathcal{O}_\lambda$. Also $\lambda(\pi_v^h)$ differs from $\lambda_v(\alpha)$ only by roots of unity in $\mathcal{O}_\lambda^*$. This shows that $(\lambda\overline{\lambda})_v^h(\pi_v)\mathcal{O}_\lambda = (\pi_v\overline{\pi}_v)^{-h(m+n)}\mathcal{O}_\lambda$. We deduce $(\lambda\overline{\lambda})_v(\pi_v) = \mathrm{Nm}(\pi_v)^{-(m+n)}$. □

Lemma 3.15. *For $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$ and $m+n$ even let χ_0 be the unramified Hecke character of infinity type $z^m\overline{z}^n$ that descends to a trivial character on the class group of F. Then χ_0 is anticyclotomic.*

Proof. We first show existence and uniqueness: Since $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$ we have $F^* \cap \prod_v \mathcal{O}_v^* = \mathcal{O}^* = \{\pm 1\}$. Since $(-1)^{m+n} = 1$, χ_0 therefore is well-defined on $F^* \cdot \mathbf{C}^* \prod_v \mathcal{O}_v^*$. The additional condition of χ_0 being trivial on $F^*\backslash\mathbf{A}_{F,f}^*/\prod_v \mathcal{O}_v^*$ determines uniquely the character on $\mathbf{A}_F^*$.

Interpreting χ_0 as a character on ideals we need to show that $\chi_0(\overline{\mathfrak{p}}) = \overline{\chi}_0(\mathfrak{p})$ for all prime ideals $\mathfrak{p}$ of F. This follows from the preceding lemma and $\chi_0(\mathfrak{p})\chi_0(\overline{\mathfrak{p}}) = \chi_0((\mathrm{Nm}(\mathfrak{p}))) = \mathrm{Nm}(\mathfrak{p})^{-(m+n)}$. □

Lemma 3.16. *Any unramified Hecke character λ of an imaginary quadratic field F is anticyclotomic.*

Proof. We first note that finite order unramified characters, i.e., characters with trivial infinity type, descend to characters on the ideal class group

$$\mathrm{Cl}(F) \cong F^*\backslash \mathbf{A}_F^*/\mathbf{C}^* \prod_v \mathcal{O}_v^*,$$

and are therefore anticyclotomic since $[\overline{\mathfrak{a}}] = [\mathfrak{a}]^{-1}$ in $\mathrm{Cl}(F)$.

For $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$ if the unramified character λ has infinity type $\lambda_\infty(z) = z^m\overline{z}^n$ then $m+n$ is even and we let χ_0 be as in the preceding lemma. We consider now λ/χ_0. Since this is a finite order unramified character we deduce that λ is anticyclotomic.

For $F = \mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ we use that their class number is one: This implies that the finite order unramified character $\frac{\overline{\lambda}(\overline{x})}{\lambda(x)}$ is trivial.

□

Example 3.17. Our main theorem will demand unramified characters χ with infinity type z^2. The proof of Lemma 3.16 shows that all such characters are given by composition of characters on the ideal class group with χ_0.

We will have some freedom in how to factor χ as μ_1/μ_2 (where $(\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ is then used in the definition of the Eisenstein cohomology class). At some point it will be necessary to find such μ_i that are anticyclotomic. For this the following result by Greenberg in [G85] will be useful:

Lemma 3.18. *Let F be an arbitrary imaginary quadratic field. Then there exists an anticyclotomic Grössencharacter μ_G of infinity type z^{-1} whose conductor is divisible precisely by the ramified primes of F. If $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$ its restriction to the units at the ramified places can be taken to have order 2.*

Remark 3.19. We will frequently use the inverse of this character (which is also anticyclotomic). An important property of these characters is that, for F different from $\mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$, their conductors contain a prime with respect to which -1 and 1 are non-congruent (i.e., some prime with residue characteristic $p \geq 3$).

Proof. For $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$ Greenberg takes the Größencharaktere associated to specific elliptic curves. In the other cases he uses the same method of construction as in Lemma 3.15 (i.e., extension to $\mathbf{A}_F^*$ from $F^* \prod_v \mathcal{O}_v^* \mathbf{C}^*$). At the ramified places v the character is defined to be trivial on the local norm of the units (viewed as a subgroup of $\mathcal{O}_v^*$). We refer to [G85] for the proof, especially that this character is anticyclotomic. □

The following character will be useful because of its minimal ramification (see also [Ti] Lemme 2.5):

Lemma 3.20. *Let $\ell \geq 5$ be a rational prime and $\mathfrak{l}$ a prime of F dividing ℓ. Then there exists a Hecke character $\mu^{\mathfrak{l}}$ with conductor $\mathfrak{l}$ of infinity type z.*

Proof. Since $\ell \geq 5$, $\mathfrak{l}$ separates the roots of unity and so the character is well-defined on $F^* \cdot \mathbf{C}^* U(\mathfrak{l})$. Since the ray class group $F^* \backslash \mathbf{A}_{F,f}^* / U(\mathfrak{l})$ is finite we can trivially extend to a continuous character on $\mathbf{A}_F^*$. □

We will be using the following properties of the values of algebraic Hecke characters: Suppose $\mathfrak{p}$ is a prime in the imaginary quadratic field F. Denote the underlying rational prime by p.

Lemma 3.21. *For $\mu : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ with infinity type $z^a \overline{z}^b$ for $a, b \in \mathbf{Z}$ we let $\mathcal{O}_\mu$ denote the ring of integers in the finite extension F_μ of $F_\mathfrak{p}$ obtained by adjoining the values of the finite part of μ. Denote by v_0 the place corresponding to $\mathfrak{p}$.*

(1) *For $x \in \mathbf{A}_F^*$ with $x_{v_0} \in \mathcal{O}_{v_0}^*$ and $x_{\overline{v}_0} \in \mathcal{O}_{\overline{v}_0}^*$ we have $\mu(x) \in \mathcal{O}_\mu^*$.*

(2) *If* $\begin{cases} a \geq 0 \\ a \leq 0 \end{cases}$ *then* $\begin{cases} \mu^{-1}(\pi_{v_0}) \in \mathcal{O}_\mu \\ \mu(\pi_{v_0}) \in \mathcal{O}_\mu \end{cases}$.

Proof. Since μ has finite order on $\mathcal{O}_v^*$ it suffices for (1) to show that $\mu(\pi_w) \in \mathcal{O}_\mu^*$ for $w \neq v_0, \overline{v}_0$. Denote the prime ideal corresponding to π_w by $\mathcal{P}_w$. If h is the class number of F, we have $\mathcal{P}_w^h = (\alpha)$ for $\alpha \in \mathcal{O}$ and $\alpha \in \mathcal{O}_v^*$ for all $v \neq w$. Now

$$1 = \mu((\alpha, \alpha, \ldots)) = \mu_\infty(\alpha)\mu_w(\alpha) \prod_{v \neq w} \mu_v(\alpha).$$

Since $\prod_{v\neq w} \mu_v(\alpha) \in \mathcal{O}_\mu^*$ and $\mu_\infty(\alpha) = \alpha^a \overline{\alpha}^b \in \mathcal{O}_{v_0}^*$ we deduce that $\mu_w(\alpha) \in \mathcal{O}_\mu^*$, i.e., $\mathrm{val}_{\mathfrak{p}}(\mu_w(\alpha)) = 0$, which implies $\mu(\pi_w) \in \mathcal{O}_\mu^*$.

For (2) the same argument for $w = v_0$ shows that $\mathrm{val}_{\mathfrak{p}}(\mu_w(\alpha)) = -\mathrm{val}_{\mathfrak{p}}(\mu_\infty(\alpha))$. □

3.6 Integrality and rationality results

Definition 3.22. Let p be a prime of $\mathbf{Z}$ split in the imaginary quadratic field F and let $\mathfrak{p}$ be one of the primes above it. Let $\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ be a character with $\phi \in S^1(m, n, 0, 0)$ or $\phi \in \overline{S}^1(m, n, 0, 0)$. We put $\chi := \mu_1/\mu_2$. Let $\mathcal{O}_\chi$ denote the ring of integers in the finite extension F_χ of $F_{\mathfrak{p}}$ obtained by adjoining the values of the finite part of both μ_i and $L^{\mathrm{alg}}(0, \chi) \sim \frac{L(0,\chi)}{\Omega^2}$, where Ω is a complex period depending only on F. (For the algebraicity and p-adic integrality of the special L-function value see Theorem 2.1.)

Let

$$\widetilde{H}^1(X, \widetilde{M}_{\mathcal{O}_\chi}) := \mathrm{im}(H^1(X, \widetilde{M}_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_\chi) \to H^1(X, \widetilde{M} \otimes_F F_\chi))$$

for $X \subset \overline{S}_{K_f}$. We will need the following results:

Proposition 3.23. *Assume that the conductors of μ_1 and μ_2 are coprime to (p). For constant coefficient systems (i.e., the infinity type of μ_1 equals z and of μ_2 equals z^{-1}) we have*

$$[\omega_0(\phi, \Psi_\phi)], [\omega_0(w_0.\phi, \Psi_{w_0.\phi})] \in \widetilde{H}^1(\partial \tilde{S}_{K_f^s}, \mathcal{O}_\chi).$$

(Here Ψ_ϕ and $\Psi_{w_0.\phi}$ are the functions defined in Definition 3.4.)

Proof. In Section 3.3 we made explicit the isomorphism

$$H^1(\partial \tilde{S}_{K_f^s}, R) \cong \bigoplus_{[\det(\xi)] \in H_{K^s}} \bigoplus_{[\eta] \in \mathbf{P}^1(F)/\Gamma_\xi} H^1(\Gamma_{\xi, B^\eta}, R),$$

which is functorial for $\mathcal{O}[\frac{1}{6}]$-algebras R. Here $H_{K^s} := \mathbf{A}_F^*/\det(K^s)F^*$ for $K^s := K_f^s K_\infty$, $\Gamma_\xi = G(\mathbf{Q}) \cap \xi K_f^s \xi^{-1}$, and $\Gamma_{\xi,B^\eta} = \Gamma_\xi \cap \eta^{-1}B(\mathbf{Q})\eta$. Using this description we associated to $\omega_0(\phi,\Psi_\phi)$ a group cohomology class for $R = \mathbf{C}$ in Lemma 3.10. We claim that it lies in the image of the natural map from $H^1(\Gamma_{\xi,B^\eta},\mathcal{O}_\chi)$. Analyzing the expression in Lemma 3.10 we need to show for all η and ξ and for all matrices $\begin{pmatrix} a & x \\ 0 & d \end{pmatrix} \in \Gamma_{\xi,B}$ that $x\Psi_\phi(\eta_f\xi)$ and $\overline{x}\Psi_{w_0.\phi}(\eta_f\xi)$ lie in $\mathcal{O}_\chi$. One checks that it is sufficient to prove this for a specific choice for the set of representatives ξ and η. If we can find a set of representatives ξ and η whose $\mathfrak{p}$- and $\overline{\mathfrak{p}}$-components are units (i.e., they are elements of $\mathrm{GL}_2(\mathcal{O}_p) := \prod_{v|p}\mathrm{GL}_2(\mathcal{O}_v)$) then the definition of Ψ_ϕ at places away from the conductors of the μ_i together with Lemma 3.21 shows that each $\Psi_\phi(\eta_f\xi)$ and $\Psi_{w_0.\phi}(\eta_f\xi)$ lies in $\mathcal{O}_\chi^*$. For ξ this can be achieved because H_{K^s} is a generalized ideal class group. The Chebotarev density theorem implies that each class is represented by a prime different from $\mathfrak{p}$ and $\overline{\mathfrak{p}}$ so we can choose ξ such that its $\mathfrak{p}$- and $\overline{\mathfrak{p}}$-components equal 1. This implies, in particular, that $\Gamma_\xi \cap G(\mathbf{Q}_p) \subset \mathrm{GL}_2(\mathcal{O}_p)$, so x and $\overline{x}$ from above also lie in $\mathcal{O}_\chi$.

For finding $[\eta] \in \mathbf{P}^1(F)/\Gamma_\xi = B(\mathbf{Q})\backslash G(\mathbf{Q})/\Gamma_\xi$ with $\eta \in G(\mathbf{Q}) \cap \mathrm{GL}_2(\mathcal{O}_p)$ we claim that $\mathrm{GL}_2(F_p) := \prod_{v|p}\mathrm{GL}_2(F_v)$ satisfies

$$\mathrm{GL}_2(F_p) = B_0(F)\mathrm{GL}_2(\mathcal{O}_p).$$

The Iwasawa decomposition implies $\mathrm{GL}_2(F_p) = B_0(F_p)\mathrm{GL}_2(\mathcal{O}_p)$. By the Chinese Remainder Theorem we get $F \cdot \prod_{v|p}\mathcal{O}_v = \prod_{v|p}F_v$ and $F^* \cdot \prod_{v|p}\mathcal{O}_v^* = \prod_{v|p}F_v^*$ and so

$$B_0(F_p) = B_0(F)B_0(\mathcal{O}_p).$$

Together these prove our claim. Applying the claim to $\eta' \in \mathrm{GL}_2(F)$ one gets a decomposition $\eta' = bk$ with $b \in B_0(F)$ and $k \in \mathrm{GL}_2(\mathcal{O}_p)$. Then $[\eta'] = [k]$ with $k \in G(\mathbf{Q}) \cap \mathrm{GL}_2(\mathcal{O}_p)$.

□

Lemma 3.24. *Assume $m = n = 0$, that the conductors of μ_1 and μ_2 are coprime to (p), and that the conductor of $\chi = \mu_1/\mu_2$ is coprime to the discriminant of F.*

Assume further that at each place occuring in the conductor only one of the μ_i is ramified. Then we have either

$$[\mathrm{res}(\mathrm{Eis}(\omega_0(\phi, \Psi_\phi))] \in \widetilde{H}^1(\partial \tilde{S}_{K_f^s}, \mathcal{O}_\chi)$$

or

$$[\mathrm{res}(\mathrm{Eis}(\omega_0(w_0.\phi, \Psi_{w_0.\phi}))] \in \widetilde{H}^1(\partial \tilde{S}_{K_f^s}, \mathcal{O}_\chi).$$

If $\chi^c = \overline{\chi}$ (where $\chi^c(x) := \chi(\overline{x})$), then the two constant terms differ only by a p-adic unit and are both integral.

Remark 3.25. Given χ with conductor coprime to (p) and the discriminant of F one can always factor $\chi = \mu_1/\mu_2$ with $\mu_1 := \mu^{\mathfrak{l}}$ for a prime $\mathfrak{l}$ not dividing the conductor of χ and coprime to (p), where $\mu^{\mathfrak{l}}$ is the character defined in Lemma 3.20. Then μ_1 and $\mu_2 = \mu^{\mathfrak{l}}/\chi$ satisfy the conditions in the Lemma above.

Proof. By Proposition 3.5 the first constant term in the lemma is

$$\omega_0(\phi, \Psi_\phi) - \frac{2\pi}{\sqrt{d_F}} \cdot \frac{L(-1,\chi)}{L(0,\chi)} \cdot \omega_0(w_0.\phi, \Psi_{w_0.\phi})$$

and we put $C(\chi) := \frac{2\pi}{\sqrt{d_F}} \cdot \frac{L(-1,\chi)}{L(0,\chi)}$. The second constant term is

$$\omega_0(w_0.\phi, \Psi_{w_0.\phi}) - \frac{2\pi}{\sqrt{d_F}} \cdot \frac{L(-1,\chi^{-1}|\cdot|^2)}{L(0,\chi^{-1}|\cdot|^2)} \cdot \omega_0(\phi, \Psi_\phi).$$

Applying the functional equation (see Section 2.6) and using $\chi\overline{\chi} = |\cdot|^2_{\mathbf{A}_F}$ (see Lemma 3.14) one checks that the second constant term equals

$$\omega_0(w_0.\phi, \Psi_{w_0.\phi}) - C(\chi)^{-1} \cdot \omega_0(\phi, \Psi_\phi).$$

We note that

$$C(\chi) = \frac{2\pi}{\sqrt{d_F}} \cdot \frac{L(0,\chi|\cdot|^{-1})}{L(0,\chi)} \sim \frac{L^{\mathrm{alg}}(0,\chi|\cdot|^{-1})}{L^{\mathrm{alg}}(0,\chi)},$$

where '$\sim$' indicates equality up to the p-adic units given by the Euler factors $(1 - \lambda(\overline{\mathfrak{p}}))(1 - \lambda^*(\overline{\mathfrak{p}}))$ for $\lambda = \chi$ and $\chi|\cdot|^{-1}$. Hence $C(\chi)$ lies in F_χ by Theorem 2.1. Since either $C(\chi) \in \mathcal{O}_\chi$ or $C(\chi)^{-1} \in \mathcal{O}_\chi$, the first statement of the Lemma follows from Proposition 3.23.

For the statement for anticyclotomic characters we first observe that by the functional equation

$$C(\chi) = u(\chi) \cdot \frac{L(0, \overline{\chi})}{L(0, \chi)}$$

for

$$u(\chi) := \frac{(\mathrm{Nm}(\mathfrak{f}_\chi))^{-1/2}}{W(\tilde{\chi})}$$

with $\mathfrak{f}_\chi$ the conductor of χ and $\tilde{\chi} = \chi|\cdot|^{-1}$. If $\chi^c = \overline{\chi}$, then $L(0, \overline{\chi}) = L(0, \chi^c) = L(0, \chi)$, so $\frac{L(0,\overline{\chi})}{L(0,\chi)} = 1$. It remains to check that $u(\chi) \in \mathcal{O}_\chi^*$. Since the conductor of χ is coprime to $\mathfrak{p}$ we only need to analyze the root number $W(\tilde{\chi})$ (see Section 2.6 for its definition). The Gauss sums are p-integral because $\mathfrak{f}_\chi$ is coprime to $\mathfrak{p}$. (This uses Lemma 3.21.) Since $\mathfrak{f}_\chi$ is coprime to the discriminant (which equals $\mathrm{Nm}(\mathcal{D})$ for the different $\mathcal{D}$) we also get that the factors $\chi(\mathcal{D}_v^{-1})$ of the root number lie in $\mathcal{O}_\chi^*$.

We want to record here that for unramified characters χ one has $u(\chi) = 1$. This follows from $\tilde{\chi}(\mathcal{D}^{-1}) = -1$, which is a consequence of F being imaginary quadratic (see [Ha79] p.73). □

Proposition 3.26.

$$[\mathrm{Eis}(\omega_0(\phi, \Psi_\phi))] \in H^1(S_{K_f^s}, \widetilde{M}^{m,n} \otimes_F F_\chi).$$

References. [HaGL2] Theorem 2, Corollary 4.2.1 and [F] Satz 1.5, [S02a] Lemma 5.1(iv). The proof uses the "Multiplicity one Theorem" for automorphic forms and Lemma 2.7 (vanishing of residual interior cohomology). □

CHAPTER IV

Denominator of the Eisenstein cohomology class

For $\phi = (\mu_1, \mu_2)$ contributing to the boundary cohomology $H^1(\partial\overline{S}_{K_f}, \widetilde{M^{m,n}_{\mathbf{C}}})$ and $\Psi \in V^{K_f}_{\phi|\alpha|^{z/2}}$ we defined in Section 3.3, following Harder, the Eisenstein cohomology 1-form $\mathrm{Eis}(\Psi) := \mathrm{Eis}(\phi_f|\alpha|_f^{z/2}, \Psi)$ whose cohomology class we denote by

$$[\mathrm{Eis}(\Psi)] \in H^1(S_{K_f}, \widetilde{M^{m,n}_{\mathbf{C}}}).$$

Harder showed that this class is, in fact, F_χ-rational, i.e., lies in $H^1(S_{K_f}, \widetilde{M^{m,n}_{F_\chi}})$ (see Proposition 3.26). Recall the definition of $\mathcal{O}_\chi$ and F_χ from Definition 3.22.

Definition 4.1. If $\mathcal{O}_L$ is the ring of integers in a (local) field L, define for any $c \in H^1(S_{K_f}, L)$ the denominator (ideal)

$$\delta(c) = \{a \ \in \mathcal{O}_L : ac \in \tilde{H}^1(S_{K_f}, \mathcal{O}_L)\}.$$

(Recall that $\tilde{H}^1(S_{K_f}, \mathcal{O}_L) = \mathrm{im}(H^1(S_{K_f}, \mathcal{O}_L) \to H^1(S_{K_f}, L)$.)

We will give a lower bound on the denominator of the Eisenstein cohomology class by integrating $\mathrm{Eis}(\Psi)$ against a suitable cycle. The (relative) cycle we use is motivated by the classical modular symbol: we integrate along the path

$$\sigma : \mathbf{R}_{>0} \to \mathbf{H}_3$$

$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},$$

or rather a sum of such paths, one for each connected component of $\overline{S}_{K_f}$. An explanation of how to interpret this relative cycle and why the integral along it provides a lower bound on the denominator of classes vanishing at the 0- and ∞-cusps is given in Section 4.4.

This is the most computationally involved chapter. Instead of evaluating the integral for the Eisenstein cohomology class defined using the function Ψ_ϕ in V_ϕ we introduced in Chapter III, we consider instead the class defined using the function $\Psi^{\text{new}} \in V_\phi$ given by newvectors at all places. In 4.1 we analyze how to translate between these classes. We then calculate in 4.2 the "toroidal" integral for $\text{Eis}(\Psi^{\text{new}})$, from which the integral for a third cohomology class $\text{Eis}(\Psi''_\phi)$ easily follows. The denominator of this last class is then related to the denominator of $\text{Eis}(\Psi_\phi)$.

After restricting to constant coefficients systems (i.e., the infinity type of (μ_1, μ_2) is (z, z^{-1})) the result of the toroidal integral calculation, up to units in $\mathcal{O}_\chi$, is

$$\int_\sigma \text{Eis}(\Psi''_\phi) \sim \frac{L(0,\mu_1)L(0,\mu_2^{-1})}{L(0,\chi)} = \frac{L^{\text{alg}}(0,\mu_1)L^{\text{alg}}(0,\mu_2^{-1})}{L^{\text{alg}}(0,\chi)} \in F_\chi.$$

From this we would like to conclude that multiplication by at least $L^{\text{alg}}(0,\chi)$ is necessary to make our Eisenstein cohomology class integral, i.e., so that it lies in $\widetilde{H}^1(S_{K_f}, \mathcal{O}_\chi)$. However, to conclude this we need to control the p-adic properties of the numerator.

To extract $L^{\text{alg}}(0,\chi)$ as the bound we use results by Hida and Finis on the non-vanishing modulo p of the L-values $L^{\text{alg}}(0, \theta\mu_i^{\pm 1})$ as θ varies in an anticyclotomic $\mathbf{Z}_q$-extension for $q \neq p$. In Section 4.3 we replace $\text{Eis}(\Psi''_\phi)$ by a "twisted" version $\text{Eis}^\theta(\Psi''_\phi)$ for a finite order character θ such that $a \cdot \text{Eis}^\theta(\Psi''_\phi)$ is integral if $a \cdot \text{Eis}(\Psi''_\phi)$ is. Up to units the result of the toroidal integral is then

$$\int_\sigma \text{Eis}^\theta(\Psi''_\phi) \sim \frac{L^{\text{alg}}(0,\mu_1\theta)L^{\text{alg}}(0,\mu_2^{-1}\theta^{-1})}{L^{\text{alg}}(0,\chi)}.$$

By Hida and Finis there exists a character θ such that the numerator is a p-adic unit. The interpretation in Section 4.4 of the toroidal integral as the evaluation pairing on a relative cycle shows that the ideal generated by $L^{\text{alg}}(0,\chi)$ gives a lower bound on the denominator of $\mathrm{Eis}^\theta(\Psi''_\phi)$ and hence of $\mathrm{Eis}(\Psi''_\phi)$. In Theorem 4.17 we finally get a lower bound on the denominator of $\mathrm{Eis}(\Psi_\phi)$ in terms of $L^{\mathrm{alg}}(0,\chi)$.

4.1 Translation between newvector and spherical functions

Since we will switch between different compact open subgroups $K_f \subset G(\mathbf{A}_F)$ in this chapter, we introduce a slight modification of (2.9): For $\eta = (\eta_1, \eta_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ let

$$V_{\eta_f} = \left\{ \Psi : G(\mathbf{A}_f) \to \overline{F} \;\middle|\; \begin{array}{c} \Psi(bg) = \eta_f(b)\Psi(g), \Psi(gk) = \Psi(g) \\ \forall k \text{ in some compact open } K_f \subset G(\mathbf{A}_f) \end{array} \right\}. \tag{4.1}$$

Definition 4.2. (a) We recall from Chapter III the definition of $\Psi_\eta \in V_{\eta_f}$ (see Definition 3.4): Denote by S the finite set of places where both η_i are ramified, but η_1/η_2 is unramified. Then Ψ_η is given at the finite places by $\prod_{v\notin S}\Psi_v^{\mathrm{new}}\prod_{v\in S}\Psi_v^0$, where the Ψ_v^{new} are newvectors in the induced representation V_{ϕ_v}, and the Ψ_v^0 are spherical functions. Their definitions were as follows:

$$\Psi_v^{\mathrm{new}}(g) = \begin{cases} \eta_{1,v}(a)\eta_{2,v}(d) & \text{if } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} k, k \in K^1(\mathfrak{P}_v^s) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{P}_v^r \parallel \mathfrak{M}_1$ and $\mathfrak{P}_v^s \parallel \mathfrak{M}_1\mathfrak{M}_2$, and

$$\Psi_v^0(g) = \eta_{1,v}(a)\eta_{2,v}(d)\eta_1(\det(k)) \text{ for } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, k \in \mathrm{GL}_2(\mathcal{O}_v).$$

Note that Ψ_{η_f} lies in $V_{\eta_f}^{K_f^s}$ for $K_f^s = \prod_{v\in S} U^1(\mathfrak{M}_{1,v})\prod_{v\notin S} K^1((\mathfrak{M}_1\mathfrak{M}_2)_v)$. Here $\mathfrak{M}_i$ is the conductor of η_i.

(b) We denote by $\Psi_{\eta_f}^{\mathrm{new}} := \prod_v \Psi_v^{\mathrm{new}}$ the function defined by the newvectors at all places. Then $\Psi_{\eta_f}^{\mathrm{new}}$ lies in $V_{\eta_f}^{K_f^{\mathrm{new}}}$ for $K_f^{\mathrm{new}} := K^1(\mathfrak{M}_1\mathfrak{M}_2)$.

The following lemmata tell us how to translate between Ψ_{η_f} and $\Psi_{\eta_f}^{\mathrm{new}}$.

Lemma 4.3. *Let v be a place where both η_i are unramified, and $\mu : F_v^* \to \mathbf{C}^*$ a character. If we denote by Ψ the newvector in V_{η_v} then $\Psi'(g) = \Psi(g)\mu(\det(g))$ is the spherical function in $V_{\eta_v\mu}$.*

Proof. Applying the Iwasawa decomposition to $g \in \mathrm{GL}_2(F_v)$ we get

$$\begin{aligned}\Psi'(g) &= \Psi'(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k) = \Psi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k)\mu(ad)\mu(\det(k)) = \\ &= \eta_{1,v}(a)\eta_{2,v}(d)\mu(a)\mu(d)\mu(\det(k)),\end{aligned}$$

as required for the spherical function. □

Lemma 4.4. *Let v be a place where both η_i are unramified, and $\mu : F_v^* \to \mathbf{C}^*$ a character with conductor $\mathfrak{P}_v^r$, $r > 0$. If we denote by Ψ' the spherical function in $V_{\eta_v\mu}$ and by Ψ^{new} the newvector in $V_{\eta_v\mu}$, then we have*

$$\begin{aligned}\Psi''(g) &:= \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)\Psi'(g \begin{pmatrix} 1 & \frac{x}{\pi_v^r} \\ 0 & 1 \end{pmatrix}) = \\ &= \mu^{-1}(-1)(\eta_2/\eta_1)(\mathfrak{P}_v^r) \cdot L_v^{-1}(\eta_1/\eta_2, 0) \cdot \Psi^{\mathrm{new}}(g).\end{aligned}$$

Proof. To simplify notation we will write q for π_v^r . We need to check (1) the correct transformation under multiplication by upper triangular matrices on the left, (2) right invariance under $K^1((q^2))$, and to show (3) that the left hand side is non-zero only on $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$. Lastly we show (4) that the value on $\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix}$ is essentially the inverse of the L-factor $L_v(\eta_1/\eta_2, 0)$.

(1)

$$\begin{aligned}\Psi''(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) &= \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)\Psi'(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g \begin{pmatrix} 1 & \frac{x}{\pi_v^r} \\ 0 & 1 \end{pmatrix}) = \\ &= \eta_1(a)\mu(a)\eta_2(d)\mu(d)\Psi''(g).\end{aligned}$$

(2) Let $k = \begin{pmatrix} 1+aq^2 & b \\ cq^2 & d \end{pmatrix} \in K^1((q^2))$. Now

$$\Psi''(gk) = \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)\Psi'(g \begin{pmatrix} 1+aq^2 & b \\ cq^2 & d \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{q} \\ 0 & 1 \end{pmatrix}).$$

Write

$$\begin{pmatrix} 1+aq^2 & b \\ cq^2 & d \end{pmatrix} = \begin{pmatrix} 1+aq^2 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & b/(1+aq^2) \\ cq^2/d & 1 \end{pmatrix}.$$

One checks that

$$\begin{pmatrix} 1 & b' \\ c'q^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x}{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1-xqc' & b'-c'x^2 \\ q^2c' & 1+xqc' \end{pmatrix}$$

and

$$\begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{xa'}{qd'} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix}.$$

Using Lemma 4.4 to write $\Psi'(g) = \Psi(g)\mu(\det(g))$ for the newvector Ψ in V_{η_v}, we obtain

$$\Psi''(gk) = \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)\Psi(g \begin{pmatrix} 1 & \frac{x(1+aq^2)}{qd} \\ 0 & 1 \end{pmatrix})\mu(\det(g))\mu(d).$$

Now changing variable from x to $\frac{x(1+aq^2)}{qd}$ we get back $\Psi''(g)$.

(3) Let us first check that Ψ'' is zero on $\begin{pmatrix} 1 & 0 \\ \pi_v^n & 1 \end{pmatrix}$ for $n \neq r$. Let

$$h = \begin{pmatrix} 1 & 0 \\ \pi_v^n & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{\pi_v^r} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x\pi_v^{-r} \\ \pi_v^n & 1+x\pi_v^{n-r} \end{pmatrix}.$$

In the case $n > r$, so $1 + x\pi_v^{n-r} \in \mathcal{O}_v^*$, we have

$$h = \begin{pmatrix} \frac{1}{1+x\pi_v^{n-r}} & x\pi_v^{-r} \\ 0 & 1+x\pi_v^{n-r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{\pi_v^n}{1+x\pi_v^{n-r}} & 1 \end{pmatrix},$$

so $\Psi'(h) = \eta_1^{-1}(1+x\pi_v^{n-r})\eta_2(1+x\pi_v^{n-r})\mu(1) = 1$, since the η_i are unramified by assumption. This means that $\Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^n & 1 \end{pmatrix}) = \sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x) = 0$. The latter follows from the definition of the conductor: Since $\mu(\epsilon) = 1$ for $\epsilon \in \mathcal{O}_v^*$ with $\epsilon \equiv 1 \mod \mathfrak{P}_v^r$, for any $\epsilon \in \mathcal{O}_v^*$ $\mu(\epsilon)$ depends only on $\epsilon \mod \mathfrak{P}_v^r$. There exists $\epsilon \in \mathcal{O}_v^*$ such that $\mu(\epsilon) \neq 1$. Multiplying by this non-trivial $\mu^{-1}(\epsilon)$ does not change the above sum, so the sum must be zero. The case $n < r$ is treated similarly.

Alternatively, we observe that (1) and (2) already force $\Psi'' \in V_{\eta_v\mu}^{K^1((q^2))}$ to be a multiple of the newvector, which is nonzero only on $\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}$.

(4) We now come to the calculation of $\Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix})$.

Before we start let us note the following: As already noted above, the non-trivial character μ on $\mathcal{O}_v^*$ descends to a non-trivial character on $(\mathcal{O}_v/\mathfrak{P}_v^r)^*$, which implies that

$$\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x) = 0.$$

In fact, $(\mathcal{O}_v/\mathfrak{P}_v^r)^*$ can be replaced by the subgroups $(1+\mathfrak{P}_v^n)/(1+\mathfrak{P}_v^r)$ for $n = 1,\ldots r-1$ if $r > 1$.

One checks that we have the Iwasawa decomposition

$$\begin{pmatrix} 1 & 0 \\ q & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{x}{q} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{x}{q} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{1+x} & -\frac{x^2}{q} \\ 0 & 1+x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \frac{q}{1+x} & 1 \end{pmatrix}.$$

(This works for all x in our sum if we avoid $x = -1$ in our choice of representatives for $x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*$; also recall that $q = \pi_v^r$.) We obtain

$$\Psi'(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) = \eta_1^{-1}(1+x)\mu^{-1}(1+x)\eta_2(1+x)\mu(1+x) = \eta_1^{-1}(1+x)\eta_2(1+x).$$

This gives

$$\Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) = \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)(\eta_2/\eta_1)(x+1).$$

Let us first treat the case $r = 1$. We have

$$\Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) = \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v)^*, (x+1) \neq \mathfrak{P}_v} \mu^{-1}(x) + \mu^{-1}(\pi_v - 1)(\eta_2/\eta_1)(\pi_v).$$

Using that $\sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v)^*} \mu^{-1}(x) = 0$ this equals

$$\mu^{-1}(\pi_v - 1)\left((\eta_2/\eta_1)(\pi_v) - 1\right) = \mu^{-1}(-1)(\eta_2/\eta_1)(\mathfrak{P}_v) \cdot L_v^{-1}(\eta_1/\eta_2, 0).$$

For $r > 1$ we have

$$\Psi'(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) = \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*, (x+1) \notin \mathfrak{P}_v} \mu^{-1}(x) + \sum_{x \in \mathcal{O}_v/\mathfrak{P}_v^{r-1}} \mu^{-1}(x\pi_v - 1)(\eta_2/\eta_1)(x\pi_v).$$

We rewrite the second sum as follows:

$$\sum_{x \in \mathcal{O}_v/\mathfrak{P}_v^{r-1}} \mu^{-1}(x\pi_v - 1)(\eta_2/\eta_1)(x\pi_v) =$$

$$= \mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^r) + \mu^{-1}(-1)\sum_{n=1}^{r-1}(\eta_2/\eta_1)(\pi_v^n) \sum_{u \in (\mathcal{O}_v/\mathfrak{P}_v^{r-n})^*} \mu^{-1}(1 + \pi_v^n u).$$

Let $S_n := \sum_{u \in (\mathcal{O}_v/\mathfrak{P}_v^{r-n})^*} \mu^{-1}(1 + \pi_v^n u)$. Now we make the following observation: Since $\sum_{y \in 1 + \mathfrak{P}_v^m} \mu^{-1}(y) = 0$ for $m = 1, \ldots, r-1$ by our initial remark we have

$$S_n = \sum_{w \in \mathcal{O}_v/\mathfrak{P}_v^{r-n}} \mu^{-1}(1 + \pi_v^n w) - \sum_{w \in \mathcal{O}_v/\mathfrak{P}_v^{r-n-1}} \mu^{-1}(1 + \pi_v^{n+1} w) = \begin{cases} 0 & \text{if } n \leq r-2, \\ -1 & \text{if } n = r-1. \end{cases}$$

We are left to evaluate

$$\begin{aligned} \Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) &= \left(\sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*, (x+1) \notin \mathfrak{P}_v} \mu^{-1}(x) \right) \\ &+ \mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^r) - \mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^{r-1}). \end{aligned}$$

The sum in brackets turns out to be zero as well, since

$$0 = \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x) = \sum_{x \not\equiv -1 \mod \mathfrak{P}_v} \mu^{-1}(x) + \mu^{-1}(-1) \sum_{n=1}^{r-1} S_n + \mu^{-1}(-1) =$$

$$= \sum_{x \not\equiv -1 \mod \mathfrak{P}_v} \mu^{-1}(x),$$

by our calculations above.

We conclude that

$$\Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) = \mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^r) \cdot (1 - (\eta_1/\eta_2)(\pi_v)),$$

as desired.

□

4.2 The toroidal integral

4.2.1 Definition of relative cycles

For each $K_f \subset G(\mathbf{A}_f)$ the adelic symmetric space S_{K_f} has several connected components. In fact, strong approximation implies that the fibers of the determinant map

$$S_{K_f} = G(\mathbf{Q})\backslash G(\mathbf{A})/(K_f K_\infty) \twoheadrightarrow H_K := F^*\backslash \mathbf{A}_F^*/\det(K)$$

are connected.

Any $\xi \in G(\mathbf{A}_f)$ gives rise to an injection $j_\xi : G_\infty \to G(\mathbf{A})$ with $j_\xi(g_\infty) = (g_\infty, \xi)$ and, after taking quotients, to a component $\Gamma_\xi\backslash G_\infty/K_\infty \to G(\mathbf{Q})\backslash G(\mathbf{A})/K$, where $\Gamma_\xi := G(\mathbf{Q}) \cap \xi K_f \xi^{-1}$. (Since no confusion should arise we denote both maps by j_ξ.)

Choose a system of representatives $\{[\xi] \in G(\mathbf{A}_f)\}$ for H_K. For each of these let σ_ξ be the following map:

$$\sigma_\xi = j_\xi \circ \tau : \mathbf{C}^* \to S_{K_f},$$

where $\tau : \mathbf{C}^* \to G_\infty : z \mapsto \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$. (We will also use σ_ξ to denote both a map to $G(\mathbf{A})$ and the induced map to S_{K_f}.) We will consider the path in S_{K_f} given by $\sigma_\xi|_{\mathbf{R}^*_{>0}}$, which is the restriction of a path (also denoted by σ_ξ) in $\overline{S}_{K_f}$: for each component $\mathbf{H}_3 = \mathbf{R}_{>0} \times \mathbf{C}$ that path is $\sigma_\xi : [0,\infty] \mapsto \overline{\mathbf{H}}_3 : t \mapsto (t, 0)$.

4.2.2 Calculation of the toroidal integral for Ψ^{new}

Let $\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ with $\phi \in S_1(m,n,0,0)$ (cf. Section 2.10.1). We consider now the following "toroidal integral" of $\Psi^{\text{new}} := \Psi^{\text{new}}_{\phi_f|\alpha|_f^{z/2}} \in V^{K_f^{\text{new}}}_{\phi_f|\alpha|_f^{z/2}}$ over the sum of these $h_{K^{\text{new}}} := \# H_{K^{\text{new}}}$ relative cycles (this extends the calculation of [Ko] §4.5 for $\mathbf{Q}(i)$ and just one connected component):

$$\sum_{[\xi]\in H_{K^{\text{new}}}} \int_{\sigma_\xi} \text{Eis}(\Psi^{\text{new}}) = \sum_{[\xi]\in H_{K^{\text{new}}}} \int_0^\infty \text{Eis}(\sigma_\xi(t), \Psi^{\text{new}})(d\sigma_\xi(t\frac{\partial}{\partial t}))\frac{dt}{t}.$$

Using the correspondence of Section 2.9.3 we rewrite the integrand as a relative Lie algebra cocycle (now with $\sigma_\xi : \mathbf{C}^* \to G_\infty$):

$$\sum_{[\xi]\in H_{K^{\text{new}}}} \int_0^\infty \text{Eis}(\sigma_\xi(t), \Psi^{\text{new}})(d\sigma_\xi(\frac{\partial}{\partial t}|_{t=1}))\frac{dt}{t}.$$

Using the K_∞-invariance of the Eisenstein cocycle, the argument on p.107/8 in [Ko] shows that this equals

$$\sum_{[\xi]\in H_{K^{\text{new}}}} \int_0^{2\pi}\int_0^\infty \text{Eis}(\sigma_\xi(u), \Psi^{\text{new}})(\text{Ad}(\sigma_\xi(e^{-i\varphi}))d\sigma_\xi(\frac{\partial}{\partial t}|_{t=1}))\frac{dt}{t} \wedge \frac{d\varphi}{2\pi}$$

with $u = te^{i\varphi} \in \mathbf{C}^*$. Since $d\sigma_\xi(\frac{\partial}{\partial t}|_{t=1}) = \frac{H}{2}$, this equals

$$\sum_{[\xi]\in H_{K^{\text{new}}}} \int_{\mathbf{C}^*} \text{Eis}(\sigma_\xi(u), \Psi^{\text{new}})(\frac{H}{2})\frac{i}{4\pi}\frac{du\wedge d\overline{u}}{u\overline{u}} =$$

$$= \sum_{[\xi]\in H_{K^{\text{new}}}} \int_{\mathbf{C}^*} \text{Eis}(\begin{pmatrix} 1 & 0 \\ 0 & \xi x_\infty \end{pmatrix}, \Psi^{\text{new}})(\frac{H}{2})d^*x_\infty$$

with $d^*x_\infty := \frac{i}{4\pi}\frac{du\wedge d\overline{u}}{u\overline{u}}$.

Since $K_f^{\text{new}} = K^1(\mathfrak{M}_1\mathfrak{M}_2)$ we have $\begin{pmatrix} 1 & 0 \\ 0 & x_f \end{pmatrix} \in K_f^{\text{new}}$ for $x_f \in \hat{\mathcal{O}}^*$ and $\det(K_f^{\text{new}}) = \hat{\mathcal{O}}^*$, so $H_{K^{\text{new}}} \cong \text{Cl}(F)$. This allows us to change this to an integral over $F^*\backslash \mathbf{A}_F^*$. If we normalize our measure $d^*x = d^*x_\infty \prod_{v\nmid\infty} d^*x_v$ on $\mathbf{A}_F^*$ such that for finite places v, $\int_{\mathcal{O}_v^*} d^*x_v = 1$, then we can rewrite the above sum as

$$\sum_{[\xi]\in H_{K^{\mathrm{new}}}} \int_{\hat{\mathcal{O}}^*}\int_{\mathbf{C}^*} \mathrm{Eis}(\begin{pmatrix} 1 & 0 \\ 0 & \xi x_f x_\infty \end{pmatrix}, \Psi^{\mathrm{new}})(\frac{H}{2}) d^*x_\infty d^*x_f.$$

Since $\det(K) = \mathbf{C}^*\hat{\mathcal{O}}^*$ we now recognize this as

$$\int_{F^*\backslash\mathbf{A}_F^*} \mathrm{Eis}(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \Psi^{\mathrm{new}})(\frac{H}{2}) d^*x.$$

Proposition 4.5. *Let S be the finite set of places where both μ_i are ramified, but μ_1/μ_2 is unramified. Then for* $\mathrm{Re}(z) \geq 0$

$$\sum_{[\xi]\in H_{K^{\mathrm{new}}}} \int_{\sigma_\xi} \mathrm{Eis}(\Psi^{\mathrm{new}}) = \int_{F^*\backslash\mathbf{A}_F^*} \mathrm{Eis}(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \Psi^{\mathrm{new}}_{\phi_f|\alpha|_f^{z/2}})(\frac{H}{2}) d^*x$$

converges and the value is

$$\frac{L(\mu_1, z/2)L(\mu_2^{-1}, z/2)}{L^S(\mu_1/\mu_2, z)} \cdot \left(\mu_2^{-1}(\mathfrak{M}_1)\mathrm{Nm}(\mathfrak{M}_1)^{-z/2}\right) \cdot \frac{1}{2}\frac{\Gamma(z/2+1)\Gamma(z/2+1)}{\Gamma(z+2)},$$

where $\mathfrak{M}_1$ is the conductor of μ_1. (Here the factor $\mu_2^{-1}(\mathfrak{P}_v^r)$ at places $v \in S$ stands for $\mu_{2,v}^{-r}(\pi_v)$ for our choice of uniformizer π_v in the definition of the newvector.)

Remark 4.6. We evaluate the integral here only in the case of the constant coefficient system $M^{m,n}$ with $m = n = 0$ and $\phi \in S_1(0,0,0,0)$, i.e. $\phi_\infty = (z, z^{-1})$, but [Ko] gives all the calculations necessary for the general case.

Proof. (Reference: [Ko] §4.5, [Ha02] pp.27-30)

We start by unfolding the Eisenstein series $\mathrm{Eis}(g, \Psi^{\mathrm{new}}) = \sum_{\gamma\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \Psi^{\mathrm{new}}(\gamma g)$ for $\mathrm{Re}(z) \gg 0$ and use analytic continuation to deduce the result for all z for which the integral converges.

Following [Ko] we do not use Bruhat decomposition as we did in the constant term calculation, but choose representatives for $B(\mathbf{Q})\backslash G(\mathbf{Q})$ according to the orbits of the $T(\mathbf{Q})$-action:

$$G(\mathbf{Q}) = B(\mathbf{Q}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cup B(\mathbf{Q}) w_0 \cup B(\mathbf{Q}) \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} T_1(\mathbf{Q}),$$

where $T_1(\mathbf{Q}) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} : b \in F^* \right\}$.

If we decompose the integral according to this sum, the integral over the first two summands vanishes, since $\omega_z(g_f b_\infty, \phi, \Psi^{\mathrm{new}}) = \Psi_f^{\mathrm{new}}(g_f)\omega_\infty(b_\infty)$ is zero along H (here we factor (3.1) as $\omega_z(g, \phi, \Psi) = \omega_\infty(g_\infty) \cdot \Psi(g_f))$. We would like to write the remaining term as

$$\int_{\mathbf{A}_F^*} \Psi^{\mathrm{new}}(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x_f \end{pmatrix}) \cdot \omega_\infty(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x_\infty \end{pmatrix})(\frac{H}{2}) d^* x.$$

This step is justified if the latter integral converges absolutely. Since the integrand decomposes by definition as a product of local functions, the integral can be written as a product of local integrals:

$$\prod_{v \nmid \infty} \int_{F_v^*} \Psi_v^{\mathrm{new}}(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x_v \end{pmatrix}) d^* x_v \times \int_{\mathbf{C}^*} \omega_\infty(\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & x_\infty \end{pmatrix})(\frac{H}{2}) d^* x_\infty.$$

Recalling the notation of r and s from the newvector definition in Section 3.2, we will treat the local integrals according to the following cases:

(1) v finite place, both μ_i unramified, i.e., $r = s = 0$

(2) v finite place, μ_1 ramified, μ_2 unramfied, i.e., $r = s > 0$

(3) v finite place, μ_1 unramified, μ_2 ramified, i.e., $r = 0$, $s > 0$

(4) v finite place, $r > 0$ and $s - r > 0$

(5) v archimedean

Before we start, we work out the Iwasawa decomposition of our argument at the finite places:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & x_v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & x_v \end{pmatrix} = \begin{cases} \begin{pmatrix} x_v & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & x_v \end{pmatrix} & \text{if } \mathrm{ord}_v(x_v) \geq 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & x_v \end{pmatrix}\begin{pmatrix} 1 & 0 \\ x_v^{-1} & 1 \end{pmatrix} & \text{if } \mathrm{ord}_v(x_v) < 0. \end{cases}$$

We decompose F_v^* into a disjoint union of $\pi_v^t \mathcal{O}_{F_v}^*$ for $t \in \mathbf{Z}$ and note that the measure of $\pi_v^t \mathcal{O}_v^*$ with respect to $d^* x_v$ is 1 by our normalization.

In case (1), the integrand over $\pi_v^t \mathcal{O}_v^*$ is

$$\begin{cases} \mu_{1,v}^t(\pi_v)|\pi_v|_v^{tz/2} & \text{if } t \geq 0, \\ \mu_{2,v}^t(\pi_v)|\pi_v|_v^{-tz/2} & \text{if } t < 0. \end{cases}$$

The integral therefore is given by two infinite sums

$$\begin{aligned} \sum_{t\geq 0} \mu_{1,v}^t(\pi_v)|\pi_v|_v^{tz/2} \quad &+ \quad \sum_{t>0} \mu_{2,v}^{-t}(\pi_v)|\pi_v|_v^{tz/2} \\ &= \frac{1}{1-\mu_{1,v}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2}} + \frac{\mu_{2,v}^{-1}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2}}{1-\mu_{2,v}^{-1}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2}} \\ &= \frac{1-\mu_{1,v}(\pi_v)\mu_{2,v}^{-1}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z}}{(1-\mu_{1,v}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2})(1-\mu_{2,v}^{-1}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2})} \\ &= \frac{L_v(\mu_1, z/2)L_v(\mu_2^{-1}, z/2)}{L_v(\mu_1/\mu_2, z)}. \end{aligned}$$

In case (2), the definition of the newvector Ψ_v shows that the integrand is non-zero only over $\pi_v^t \mathcal{O}_v^*$ with $t \leq -r$. The integral is now

$$\sum_{t\geq r} \mu_{2,v}^{-t}(\pi_v)|\pi_v|_v^{tz/2} = \mu_{2,v}^{-r}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-rz/2} \cdot L_v(\mu_2^{-1}, z/2).$$

For case (3) we know that Ψ_v is non-zero only on

$$\left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right] \in B(F_v)\backslash \mathrm{GL}_2(F_v)/K^1(\mathfrak{P}_v^s),$$

which can only happen if $\mathrm{ord}_v(x_v) \geq 0$. The proof of Lemma 2.1 shows that for such x_v, $\begin{pmatrix} 1 & 0 \\ 1 & x_v \end{pmatrix} = \begin{pmatrix} x_v & * \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} k$ with $k \in K^1(\mathfrak{P}_v^s)$, so the integral is

$$\sum_{t \geq 0} \mu_{1,v}^t(\pi_v)|\pi_v|_v^{tz/2} = L_v(\mu_1, z/2).$$

In case (4), Ψ_v is non-zero only on $\left[\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}\right] \in B(F_v)\backslash \mathrm{GL}_2(F_v)/K^1(\mathfrak{P}_v^s)$. This means we have to have $\mathrm{ord}_v(x_v) = -r$ exactly. If $x_v = \epsilon \pi_v^{-r}$ with $\epsilon \in \mathcal{O}_v^*$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & x_v \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x_v^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon\pi_v^{-r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \epsilon^{-1}\pi_v^r & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 \\ 0 & \epsilon\pi_v^{-r} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & \epsilon^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \pi_v^{-r} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix}.$$

The integral therefore is given by

$$\int_{\mathcal{O}_v^*} \mu_{2,v}^{-r}(\pi_v)|\pi_v|_v^{rz/2} d^*\epsilon = \mu_{2,v}^{-r}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-rz/2}.$$

In Case (5) the (archimedean) factor is

$$\frac{i}{8\pi} \int_{\mathbf{C}^*} \omega_\infty(\begin{pmatrix} 1 & 0 \\ 1 & u \end{pmatrix})(H) \frac{du \wedge d\overline{u}}{u\overline{u}}$$

(cf. [Ko] pp.111-113). Here we denote $(\phi_\infty|\alpha|_\infty^{z/2})(b_\infty)k_\infty^{-1}.\check{S}_+$ by $\omega_\infty(b_\infty k_\infty)$, where $\phi_\infty = (\mu_{1,\infty}, \mu_{2,\infty})$, so

$$\omega_\infty(b_\infty k_\infty)(H) = ((\mu_{1,\infty}, \mu_{2,\infty})|\alpha|_\infty^{z/2})(b_\infty)\check{S}_+(\mathrm{Ad}(k_\infty)(H)).$$

One checks that

$$\begin{pmatrix} 1 & 0 \\ 1 & u \end{pmatrix} = \begin{pmatrix} \frac{u}{\sqrt{1+u\overline{u}}} & \frac{1}{\sqrt{1+u\overline{u}}} \\ 0 & \sqrt{1+u\overline{u}} \end{pmatrix} \begin{pmatrix} \frac{\overline{u}}{\sqrt{1+u\overline{u}}} & -\frac{1}{\sqrt{1+u\overline{u}}} \\ \frac{1}{\sqrt{1+u\overline{u}}} & \frac{u}{\sqrt{1+u\overline{u}}} \end{pmatrix}.$$

We obtain therefore

$$\omega_\infty(\begin{pmatrix}1 & 0\\ 1 & u\end{pmatrix})(H) = (\phi_\infty|\alpha|_\infty^{z/2})(\begin{pmatrix}\frac{u}{\sqrt{1+u\overline{u}}} & \frac{1}{\sqrt{1+u\overline{u}}}\\ 0 & \sqrt{1+u\overline{u}}\end{pmatrix})\check{S}_+(\begin{pmatrix}\frac{\overline{u}}{\sqrt{1+u\overline{u}}} & -\frac{1}{\sqrt{1+u\overline{u}}}\\ \frac{1}{\sqrt{1+u\overline{u}}} & \frac{u}{\sqrt{1+u\overline{u}}}\end{pmatrix}.H) =$$

$$= \frac{u}{\sqrt{1+u\overline{u}}}\frac{1}{\sqrt{1+u\overline{u}}}\left|\frac{u}{1+u\overline{u}}\right|_\infty^{z/2}\cdot\check{S}_+(\begin{pmatrix}\frac{\overline{u}}{\sqrt{1+u\overline{u}}} & -\frac{1}{\sqrt{1+u\overline{u}}}\\ \frac{1}{\sqrt{1+u\overline{u}}} & \frac{u}{\sqrt{1+u\overline{u}}}\end{pmatrix}.H).$$

Checking the action of K_∞ on the Lie algebra (see Section 1.4), we get

$$\omega_\infty(\begin{pmatrix}1 & 0\\ 1 & u\end{pmatrix})(H) = 2\frac{(u\overline{u})^{z/2+1}}{(1+u\overline{u})^{z+2}}.$$

This gives rise to Beta-Function integrals, which converge for $\mathrm{Re}(z) > -1$. The archimedean integral therefore contributes

$$\frac{i}{4\pi}\int_{\mathbf{C}^*}\frac{(u\overline{u})^{z/2+1}}{(1+u\overline{u})^{z+2}}\frac{du\wedge d\overline{u}}{u\overline{u}} = \frac{1}{2}\frac{\Gamma(z/2+1)\Gamma(z/2+1)}{\Gamma(z+2)}.$$

The preceding analysis also shows that all the local integrals converge absolutely for $\mathrm{Re}(z) > -1$ and that their product exists so the integral over $\mathbf{A}_F^*$ converges absolutely.

To conclude the proof of the proposition by analytic continuation it suffices to prove that for any $\xi \in G(\mathbf{A}_f)$

$$\int_{\sigma_\xi}\mathrm{Eis}(\Psi^{\mathrm{new}}) = \int_0^\infty \mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix}, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}})(\frac{H}{2})\frac{dt}{t}$$

converges to a holomorphic function in z for $\mathrm{Re}(z) \geq 0$. The following argument is adapted from [S02a] Proposition 3.5 and [Wes] Proposition 2.4 and shows convergence of the integral for $\mathrm{Eis}(\Psi^{\mathrm{new}}) \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f^{\mathrm{new}})(\omega^{-1})\otimes M_{\mathbf{C}}^{m,n})$ for all m, n.

For $c > 0$ let

$$I_c(z) := \int_{1/c}^c \mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix}, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}})(\frac{H}{2})\frac{dt}{t}.$$

For any $c > 0$ this is a holomorphic function for all z with $\mathrm{Re}(z) \geq 0$ since the Eisenstein cohomology class is holomorphic for z in this region (cf. [Ha82] p. 123).

It suffices therefore to show that $I_c(z)$ converges locally uniformly for all z with $\mathrm{Re}(z) \geq 0$ as $c \to \infty$. Recall that $\mathrm{Eis}(\Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}) = \mathrm{Eis}(\omega_z(\phi, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}))$. If we write

$$\omega_z(g, \phi, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}) = (\alpha_1(g_\infty)\check{S}_+ + \alpha_2(g_\infty)\frac{\check{H}}{2} + \alpha_3(g_\infty)\check{S}_-)\Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}(g_f)$$

and let

$$(\alpha_i, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}) \in V^{K_f^{\mathrm{new}}}_{\phi|\alpha|^{z/2}} \otimes M^{m,n}_{\mathbf{C}} : (g_\infty, g_f) \mapsto \alpha_i(g_\infty)\Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}(g_f),\ i = 1, 2, 3$$

then

$$I_c(z) = \int_{1/c}^{c} \mathrm{Eis}(\alpha_2, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}})(\begin{pmatrix} 1 & 0 \\ 0 & \xi t \end{pmatrix})\frac{dt}{t}.$$

Put $E_z(g) = \mathrm{Eis}(\alpha_2, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}})(g)$. Note that the constant term $\mathrm{res}(E_z)(g)$ vanishes for $g = \begin{pmatrix} 1 & 0 \\ 0 & \xi t \end{pmatrix}$ and $g = \begin{pmatrix} \xi & 0 \\ 0 & t \end{pmatrix} w_0$ since

$$\mathrm{res}(\mathrm{Eis}(\omega_z(\phi, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}})))(g) = \omega_z(g, \phi, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}}) + d(\phi, \Psi^{\mathrm{new}}_{\phi|\alpha|^{z/2}})\omega_{-z}(g, w_0.\phi, \Psi^{\mathrm{new}}_{w_0.(\phi|\alpha|^{z/2})}),$$

which vanishes for these g on multiples of H. It follows that

$$I_c(z) = I_c^1(z) + I_c^2(z),$$

where

$$I_c^1(z) = \int_{1/c}^{1} E_z(\begin{pmatrix} 1 & 0 \\ 0 & \xi t \end{pmatrix}) - \mathrm{res}(E_z)(\begin{pmatrix} 1 & 0 \\ 0 & \xi t \end{pmatrix})\frac{dt}{t},$$

$$I_c^2(z) = \int_{1/c}^{1} t^{-(m+n)} \left(E_z(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} w_0) - \mathrm{res}(E_z)(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} w_0) \right) \frac{dt}{t}.$$

The expression for $I_c^2(z)$ follows from the identity

$$\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} = w_0^{-1} \begin{pmatrix} 1 & 0 \\ 0 & t^{-1} \end{pmatrix} w_0 \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}$$

and a change of variables.

We note that $E_z \in \mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f) \otimes M^{m,n}_{\mathbf{C}}$, where $\mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$ is the space of automorphic forms, a certain subspace of $C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$ of functions of moderate growth (see [B92] §3, [HC] IV Theorem 7). Therefore standard growth estimates for automorphic forms on Siegel sets (see [Langl] Lemma 3.4, [Schw] §1.10, [HC] I Lemma 10) imply that for any $g \in G(\mathbf{A})$ and $r \in \mathbf{R}$ there exists a constant $C(g, r, z) > 0$, locally uniform in z, such that

$$\|E_z(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} g) - \mathrm{res}(E_z)(\begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix} g)\| \leq C(g, r, z)t^r, \ 0 < t \leq 1$$

for $\|\cdot\|$ the norm on $M_{\mathbf{C}}^{m,n}$ given by the scalar product defined in Section 2.5. From this it follows that $I_c^1(z)$ and $I_c^2(z)$ converge absolutely and locally uniformly for all z with $\mathrm{Re}(z) \geq 0$ as $c \to \infty$. The limits therefore define holomorphic functions in z, as claimed above.

□

4.2.3 Calculation of the toroidal integral for a twisted version of Ψ_ϕ

As recalled at the start of this chapter we defined a particular $\Psi_{\phi|\alpha|^{z/2}} \in V_{\phi|\alpha|^{z/2}}$ for $\phi = (\mu_1, \mu_2)$, given by $\prod_{v \notin S} \Psi_v^{\mathrm{new}} \prod_{v \in S} \Psi_v^0$. Denote from now on by $\Psi''_{\phi|\alpha|^{z/2}}$ the multiple twisted sum

$$\Psi''_{\phi_f|\alpha|_f^{z/2}}(g) = \sum_{v \in S} \sum_{x \in (\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu_1^{-1}(x) \Psi_{\phi_f|\alpha|_f^{z/2}}(g \begin{pmatrix} 1 & \frac{x}{\pi_v^r} \\ 0 & 1 \end{pmatrix}_v),$$

where $\mathfrak{P}_v^r \parallel \mathrm{cond}(\mu_1) = \mathfrak{M}_1$.

Lemma 4.4 shows that $\Psi''_{\phi_f|\alpha|_f^{z/2}}$ equals Ψ^{new} up to L-factors. We conclude that the toroidal integral for the Eisenstein series $\mathrm{Eis}(\Psi''_{\phi_f|\alpha|_f^{z/2}})$ has the following value:

Lemma 4.7.

$$\sum_{[\xi] \in H_{K^{\mathrm{new}}}} \int_{\sigma_\xi} \mathrm{Eis}(\Psi''_{\phi_f|\alpha|_f^{z/2}}) =$$

$$= \int_{F^* \backslash \mathbf{A}_F^*} \mathrm{Eis}(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}, \Psi''_{\phi_f|\alpha|_f^{z/2}})(\frac{H}{2}) d^*x = \frac{L(\mu_1, z/2) L(\mu_2^{-1}, z/2)}{L(\mu_1/\mu_2, z)} \cdot$$

$$\cdot \left(\mu_2^{-1}(\mathfrak{M}_1) \mathrm{Nm}(\mathfrak{M}_1)^{-z/2}\right) \cdot \frac{1}{2} \frac{\Gamma(z/2+1)\Gamma(z/2+1)}{\Gamma(z+2)} \cdot \prod_{v \in S} \mu_{2,v}^{-1}(-1)(\mu_2/\mu_1)(\mathfrak{P}_v^r) \mathrm{Nm}(\mathfrak{P}_v^r)^z$$

4.3 Twisting by a finite character

In order to determine a bound for the denominator of the Eisenstein cohomology class, we will also need the toroidal integral for the following twisted sum: Let $\theta : F^*\backslash \mathbf{A}_F^* \to \mathbf{C}^*$ be a finite order character of prime-power conductor q^r, with q an odd prime of $\mathbf{Z}$ such that $(q, \mathfrak{M}_1\mathfrak{M}_2) = 1$, where $\mathfrak{M}_i$ is the conductor of μ_i.

Throughout this section we assume q is inert in F. The modification necessary for q split is notationally cumbersome, but all one has to do is to repeat the twisting process twice, once for each place above q.

Let $\eta := \phi|\alpha|^{z/2} : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ and $\eta =: (\eta_1, \eta_2)$. For $\Psi \in V_\eta$ put

$$\mathrm{Eis}^\theta(g, \Psi) := \sum_{x \in (\mathcal{O}_q/\mathfrak{P}_q^r)^*} \overline{\theta}_q(x) \mathrm{Eis}(g \begin{pmatrix} 1 & -x/q \\ 0 & 1 \end{pmatrix}_q, \Psi).$$

Note that

$$\mathrm{Eis}^\theta(g, \Psi) = \mathrm{Eis}(g, \Psi^\theta),$$

where $\Psi^\theta = \Psi_q^\theta(g_q) \prod_{v \neq q} \Psi_v(g_v)$ and $\Psi_q^\theta(g_q) = \sum_{x \bmod q} \overline{\theta}(x) \Psi_q(g_q \begin{pmatrix} 1 & -x/q \\ 0 & 1 \end{pmatrix}_q)$.

We can apply our analysis in section 4.1 to relate $\Psi_{\eta,q}^{\mathrm{new},\theta}$ to some newvector: Firstly, by Lemma 4.3, $\Psi'(g) := \Psi_{\eta,q}^{\mathrm{new}}(g)\theta_q(\det(g))$ is the spherical function for $V_{\eta_q\theta_q}$ (we use here that the conductors of η_i and θ are relatively prime). Lemma 4.4 tells us that $\Psi''(g) = \sum_{x \bmod q} \overline{\theta}(x) \Psi'(g \begin{pmatrix} 1 & -x/q \\ 0 & 1 \end{pmatrix}_q)$ is the newvector in $V_{\eta_q\theta_q}$, multiplied by $\theta_q^{-1}(-1)(\eta_2/\eta_1)(q^r) \cdot L_q^{-1}(\eta_1/\eta_2, 0)$. Untwisting by $\theta_q(\det(g))$ we deduce that

Lemma 4.8.

$$\Psi_{\eta,q}^{\mathrm{new},\theta}(g) = \Psi_{\eta\theta,q}^{\mathrm{new}}(g)\overline{\theta}_q(-\det(g)) \cdot (\eta_2/\eta_1)(q^r) \cdot L_q^{-1}(\eta_1/\eta_2, 0).$$

□

This implies the following:

Corollary 4.9. *$\Psi_{\eta_f}^{\mathrm{new},\theta} \in V_{\eta_f\theta}^{K_f^\theta}$ for $K_f^\theta := \prod_{v \neq q} K^1(\mathfrak{M}_{1,v}\mathfrak{M}_{2,v}) \cdot (K^1((q^r)) \cap U^1((q^r)))$ (see Section 3.2 for the definition of $U^1((q^r))$).*

The translation of our toroidal integral over copies of $\mathbf{C}^*$ to an integral over $F^*\backslash\mathbf{A}_F^*$ is now slightly more complicated, since Ψ_η^θ is not right-invariant under $\begin{pmatrix}1 & 0\\ 0 & x_q\end{pmatrix}$ for $x_q \in \mathcal{O}_q^*$. We have instead $\Psi_\eta^\theta(g\begin{pmatrix}1 & 0\\ 0 & x_q\end{pmatrix}) = \Psi_\eta^\theta(g)\overline{\theta}_q(x_q)$ by Lemma 4.8. This leads to

Lemma 4.10.

$$\sum_{[\xi]\in H_{K^\theta}} \int_{\mathbf{C}^*} \mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & \xi x_\infty\end{pmatrix}, \Psi^{\mathrm{new},\theta}_{(\mu_1,\mu_2)|\alpha|^{z/2}})(\frac{H}{2})d^*x_\infty =$$

$$= \int_{F^*\backslash\mathbf{A}_F^*} \mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}, \Psi^{\mathrm{new},\theta}_{(\mu_1,\mu_2)|\alpha|^{z/2}})\theta(x)(\frac{H}{2})d^*x.$$

Proof. We again want to replace the argument by $\begin{pmatrix}1 & 0\\ 0 & \xi x_\infty x_f\end{pmatrix}$ for $x_f \in \hat{\mathcal{O}}^*$. As we just showed, $\mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & \xi x_\infty x_f\end{pmatrix}, \Psi^{\mathrm{new},\theta}_{(\mu_1,\mu_2)|\alpha|^{z/2}}) = \mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & \xi x_\infty\end{pmatrix}, \Psi^{\mathrm{new},\theta}_{(\mu_1,\mu_2)|\alpha|^{z/2}})\overline{\theta}_q(x_q)$. Since by assumption θ is unramified away from q, $\theta_\infty = 1$, and by choosing our representatives ξ to be unramified at q we can replace $\overline{\theta}_q(x_q)$ by $\overline{\theta}(\xi x_\infty x_f)$ and hence obtain the right hand side after a change of variables. □

For $\mathrm{Re}(z) \geq 0$ the value of the integral in Lemma 4.10 is now given by

Proposition 4.11.

$$\int_{F^*\backslash\mathbf{A}_F^*} \mathrm{Eis}(\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix}, \Psi^{\mathrm{new},\theta}_{(\mu_1,\mu_2)|\alpha|^{z/2}})\theta(x)(\frac{H}{2})d^*x =$$
$$= \frac{L(\mu_1\theta, z/2)L(\mu_2^{-1}\theta^{-1}, z/2)}{L^S(\mu_1/\mu_2, z)} \cdot \frac{\Gamma(z/2+1)\Gamma(z/2+1)}{\Gamma(z+2)} \cdot$$
$$\cdot \frac{1}{2}\left((\theta\mu_2)^{-1}(\mathfrak{M}_1 q^r)\mathrm{Nm}(\mathfrak{M}_1 q^r)^{-z/2}\right) \cdot \overline{\theta}(-1)(\mu_2/\mu_1)(q^r)\mathrm{Nm}(q^r)^z,$$

where $\mathfrak{M}_1$ is the conductor of μ_1 and S is the finite set of places where both μ_i are ramified, but μ_1/μ_2 is unramified.

Proof. Away from the place q one can quickly repeat the calculations from Prop. 4.5 to see the effect of the twist by θ. At q we are looking at the integral

$$\int_{F_q^*} \Psi^{\mathrm{new},\theta}_{(\mu_1,\mu_2)|\alpha|^{z/2},q}(\begin{pmatrix} 1 & 0 \\ 1 & x_q \end{pmatrix})\theta(x_q)d^*x_q.$$

By Lemma 4.8 this equals (use $\theta|_{F^*} = 1$)

$$\overline{\theta}(-1)(\mu_2/\mu_1)(q^r)\mathrm{Nm}(q^r)^z L^{-1}(\mu_1/\mu_2, z) \cdot \int_{F_q^*} \Psi^{\mathrm{new}}_{(\mu_1,\mu_2)|\alpha|^{z/2}\theta,q}(\begin{pmatrix} 1 & 0 \\ 1 & x_q \end{pmatrix})d^*x_q.$$

Now we are in the situation of case (4) of Proposition 4.5, and we obtain

$$\overline{\theta}(-1)(\mu_2/\mu_1)(q^r)\mathrm{Nm}(q^r)^z L^{-1}(\mu_1/\mu_2, z) \cdot (\mu_2\theta)_q^{-1}(q^r)q^{-rz/2}.$$

□

4.4 Relative cohomology and homology

4.4.1 Definitions

Let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic subgroup and M an $\mathcal{O}[\Gamma]$-module. We define the homology $H_i(\Gamma\backslash\overline{\mathbf{H}}_3, \tilde{M})$ as the homology of the complex of Γ-coinvariants of singular chains $(C_\bullet(\overline{\mathbf{H}}_3) \otimes M)_\Gamma$.

We recall the definition of relative singular homology and cohomology (for constant coefficients R; see [B67] for the general case): For a subspace A of a manifold X define the singular i-chains $C_i(A, R)$ to be the free R-module generated by all singular i-simplices $\Delta_i \to A$ and let $C_\bullet(X, A, R) := C_\bullet(X, R)/C_\bullet(A, R)$. The relative homology $H_i(X, A, R)$ is then defined as homology of this chain complex. One checks that classes in $H_i(X, A, R)$ are represented by relative cycles, i-chains $\alpha \in C_i(X, R)$ such that $\partial\alpha \in C_{i-1}(A, R)$. Define now relative (simplicial) cohomology as homology of the chain complex $C^\bullet = \mathrm{Hom}(C_\bullet(X, A, R), R)$. This implies that relative cochains are absolute cochains (for X) vanishing on chains in A. For the corresponding definitions for sheaf cohomology and Borel-Moore homology and isomorphisms with the singular theories we refer to [B67]. We revert here to singular homology and cohomology because we want to make use of the explicit evaluation pairings between them.

Proposition 4.12 ([F], Satz 3, [G67] §23). *1. For the ring $R = \mathcal{O}[\frac{1}{6}]$ and a subspace $A \subset \Gamma\backslash\overline{\mathbf{H}}_3$ the evaluation pairing*

$$H_i(\Gamma\backslash\overline{\mathbf{H}}_3, A, \widetilde{M^\vee \otimes R})/\text{torsion} \times H^i(\Gamma\backslash\overline{\mathbf{H}}_3, A, \widetilde{M \otimes R})/\text{torsion} \to R$$

is perfect and functorial in the ring R.

2. For $R = \mathbf{C}$, the pairing between a de Rham (or relative Lie algebra) cocycle ω and a differentiable singular cycle σ is given by the integral

$$\int_\sigma \omega.$$

4.4.2 Interpretation of the toroidal integral as evaluation pairing

We have $[\mathrm{Eis}^\theta(\Psi''_\phi)] \in H^1(\overline{S}_{K_f^\theta}, \mathbf{C})$. Here $\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$. Let

$$S_{K_f^\theta} \cong \bigoplus_{[\det(\xi)] \in H_{K^\theta}} \Gamma_\xi^\theta\backslash\mathbf{H}_3.$$

The paths σ_ξ we described in 4.2.1 are not 1-cycles in $\overline{S}_{K_f^\theta}$. They are only relative cycles (cf. [Ko] §5.2) giving rise to classes in $H_1(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3, \partial(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3), \mathbf{Z})$. Since the endpoints lie in the ∞- and 0-cusps (use $\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{s} \end{pmatrix} K_\infty = w_0 \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} K_\infty$) they are, in fact, relative cycles for $H_1(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3, \Gamma_{\xi,B}^\theta\backslash e(B) \cup \Gamma_{\xi,B^w}^\theta\backslash e(B^w), \mathbf{Z})$.

If we can show that $\mathrm{Eis}^\theta(\Psi''_\phi)$ is a relative cocycle with respect to the ∞- and 0-cusps of each connected component, we can apply the evaluation pairing

$$H^1(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3, e'(B) \cup e'(B^w), R) \times H_1(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3, e'(B) \cup e'(B^w), \mathbf{Z}) \to R,$$

given by $([\omega], [\sigma_\xi]) \mapsto \int_{\sigma_\xi} \omega$ (here we follow [BS] in writing $e'(P)$ for $\Gamma_{\xi,P}^\theta\backslash e(P)$). By the functoriality in the $\mathcal{O}$-algebra R, a multiple $a\omega$ is integral only if $a([\omega], [\sigma_\xi])$ is. Our toroidal integral now corresponds to the sum of these evaluation pairings for each connected component. This allows us in the next section to deduce a lower bound on the denominator of the Eisenstein cohomology class in terms of a special L-value. We will show that the result of the toroidal integral is the inverse of the special L-value up to p-adic units.

We prove now:

Lemma 4.13. *Let F_θ be the finite extension of F_χ (see Definition 3.22) containing the values of the finite order character θ. Then we have*

$$[\mathrm{Eis}^\theta(\Psi''_\phi)] \in \bigoplus_{[\det(\xi)]\in H_{K^\theta}} H^1(\Gamma^\theta_\xi\backslash\overline{\mathbf{H}}_3, e'(B)\cup e'(B^w), F_\theta).$$

Proof. It is clear that $[\mathrm{Eis}^\theta(\Psi''_\phi)] \in H^1(\overline{S}_{K^\theta_f}, F_\theta)$. From the form of the constant term for $\mathrm{res}(\mathrm{Eis}(\omega_0(\phi,\Psi_\phi)))$ (see Proposition 3.5) we deduce, by interchanging the finite sums of the twists with the integral, that

$$\mathrm{res}(\mathrm{Eis}^\theta(\Psi''_\phi)) = \mathrm{res}(\mathrm{Eis}^\theta(\omega_0(\phi,\Psi''_\phi))) = \omega_0(\phi,(\Psi''_\phi)^\theta) + d(\phi)\omega_0(w_0.\phi,(\Psi''_{w_0.\phi})^\theta).$$

Here

$$(\Psi''_*)^\theta(g) = \sum_x \overline{\theta}(x)\Psi''_*(g\begin{pmatrix}1 & -x/q\\ 0 & 1\end{pmatrix}_q).$$

To check that the Eisenstein cocycle vanishes on 1-cycles of $e'(B)\cup e'(B^w)$ we now translate to group cohomology and homology. We showed in Lemma 3.10 that the restriction of $[\omega_0(*\cdot\theta,(\Psi''_*)^\theta)]$ to $H^1(\Gamma^\theta_{\xi,B^\eta}\backslash e(B^\eta),F_\theta)\cong H^1(\Gamma^\theta_{\xi,B^\eta},F_\theta)$ is represented by the cocycle

$$\eta^{-1}\begin{pmatrix}a & x\\ 0 & d\end{pmatrix}\eta \mapsto (\Psi''_*)^\theta(\eta_f\xi)\cdot\begin{cases}x\\ \overline{x}\end{cases},$$

the two cases depending on the infinity type of $*$.

We need to show therefore that $(\Psi''_*)^\theta$ vanishes on $\eta_f\xi$ for η the identity matrix and w_0. We note that $\xi\in G(\mathbf{A}_f)$ can be chosen to be a diagonal matrix. Then vanishing for η equal to the identity matrix follows immediately from $\sum_{x\in(\mathcal{O}_q/\mathfrak{P}_q^r)^*}\overline{\theta}(x)=0$ for the finite order character θ. For $\eta=w_0$ the vanishing follows from our definition of the newvectors Ψ_*^{new}, of which Ψ''_* is a multiple, and from our choice of q distinct from the conductors of the characters μ_1 and μ_2. □

4.4.3 Comparison with other methods

We briefly comment on other approaches to interpreting the toroidal integral. The following method is used by Harder in [Ha02] for $\mathrm{SL}_2(\mathbf{Z})$, in [Ko] §5.4 for $\mathbf{Q}(i)$, and in [Ka] §5 for $\Gamma_1(p) \subset \mathrm{SL}_2(\mathbf{Z})$: Complete the relative cycles σ_ξ (or rather their images under powers of T_v for v the place corresponding to $\mathfrak{p}$) by a chain in $H_1(\partial\overline{S}_{K_f}, \mathbf{Z})$ that is only supported on the infinity components of cusps "above 0". Then the evaluation pairing between an Eisenstein cohomology class supported only at cusps "above ∞" and the completed cycle is given by the toroidal integral. However, when these cusps coincide the calculation of the additional "boundary integral" and the bounding of its denominator is non-trivial.

Kaiser [Ka] uses a twisting argument similar to the one in the next section (but for the cycles, not the cohomology class) to deduce lower bounds for the denominator. Our approach seems to explain why he can choose cycles such that the boundary integrals vanish and provides a shorter alternative argument.

Skinner [S02a] gives a different interpretation of the toroidal integral. In [S02a] §4 he introduces a theory of "partial Borel-Serre compactifications". He reinterprets the twisted Eisenstein cocycle $\mathrm{Eis}^\theta(\Psi'')$ as a cocycle in the cohomology of the space $S_{K_f^\theta} \bigcup_\xi (e'(B) \cup e'(B^{w_0}))$, which contains the closure of the σ_ξ in $\overline{S}_{K_f^\theta}$. Since the restriction of the twisted Eisenstein cohomology class to these cusps is trivial $\mathrm{Eis}^\theta(\Psi'')$ corresponds to a class c_θ in $H^1_c(S_{K_f^\theta} \bigcup_\xi (e'(B) \cup e'(B^{w_0})), \mathbf{C})$. He shows that c_θ is again rational (in some finite extension of F_χ), and that if $a \cdot \mathrm{Eis}^\theta(\Psi'')$ is integral, then $a \cdot c_\theta$ is, too. He considers for each connected component of S_{K_f} the map of manifolds

$$\sigma_\xi|_{\mathbf{R}_{>0}} : \mathbf{R}_{>0} \to j_\xi(\Gamma_\xi \backslash \mathbf{H}_3) \subset S_{K_f}.$$

This is a proper embedding giving rise to a map

$$\sigma_\xi^* : H^1_c(S_{K_f^\theta} \bigcup_\xi (e'(B) \cup e'(B^{w_0})), R) \to H^1_c(\mathbf{R}_{>0}, R) = H^1_c(\mathbf{R}_{>0}, \mathbf{Z}) \otimes R \cong R,$$

which is functorial in R and maps c_θ to $\int_{\sigma_\xi} \mathrm{Eis}^\theta(\Psi'')$.

4.5 Bounding the denominator

Recall the definition of the denominator of a cohomology class from Definition 4.1. We are interested in bounding $\delta(\mathrm{Eis}(\Psi_\phi))$ for the Ψ_ϕ defined in Section 3.2.

From now on, we consider an odd prime p of $\mathbf{Z}$ split in F and let $\mathfrak{p} \subset \mathcal{O}$ be one of the primes dividing it. Let $\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ for μ_1 and μ_2 Hecke characters with infinity type z and z^{-1}, respectively, such that the conductors $\mathfrak{M}_i$ of μ_i are coprime to (p). Denote by $\mathcal{O}_\phi$ the ring of integers in the finite extension of $F_\mathfrak{p}$ containing the values of $\mu_{1,f}$, $\mu_{2,f}$ and $L^{\mathrm{alg}}(0, \mu_1/\mu_2)$. Here we use Theorem 2.1 on the algebraicity and integrality of the special L-value.

For a finite order character θ of prime power conductor q^r, with q a prime in $\mathbf{Z}$ distinct from p and coprime to the conductors of the μ_i, let $\mathcal{O}_\theta$ be the ring of integers in the finite extension of $\mathcal{O}_\phi$ containing the values of θ, $L^{\mathrm{alg}}(0, \mu_1\theta)$, and $L^{\mathrm{alg}}(0, (\mu_2\theta)^{-1})$ (again we use Theorem 2.1).

Observe that

$$\delta([\mathrm{Eis}(\Psi_\phi)]) \subseteq \delta([\mathrm{Eis}(\Psi''_\phi)]) \subseteq \mathcal{O}_\phi,$$

and

$$\delta([\mathrm{Eis}(\Psi''_\phi)])\mathcal{O}_\theta \subseteq \delta([\mathrm{Eis}^\theta(\Psi''_\phi)]).$$

In Section 4.4 we showed that the toroidal integral $\sum_{[\xi]\in H_{K^\theta}} \int_{\sigma_\xi} \mathrm{Eis}^\theta(\Psi''_\phi)$ gives the value of sums of evaluation pairings between relative cohomology and homology. Their functoriality in the coefficient system implies that the denominator $\delta([\mathrm{Eis}^\theta(\Psi''_\phi)])$ is bounded below by the denominator of the integral. Proposition 4.11 (for z=0) and Lemma 4.4 imply that the integral equals $\frac{L(0,\mu_1\theta)L(0,(\mu_2\theta)^{-1})}{L(0,\mu_1/\mu_2)}$, up to units in $\mathcal{O}_\theta$ (using Lemma 3.21 one checks that

$$(\theta\mu_2)^{-1}(\mathfrak{M}_1 q^r)\overline{\theta}(-1)(\mu_2/\mu_1)(q^r)\frac{1}{2}\prod_{v\in S}\mu_{2,v}^{-1}(-1)(\mu_2/\mu_1)(\mathfrak{P}_v^r) \in \mathcal{O}_\theta^*.)$$

This means that $\delta([\mathrm{Eis}^\theta(\Psi''_\phi)])$ is contained in the (possibly fractional) ideal

$$\left(\frac{L(0,\mu_1/\mu_2)}{L(0,\mu_1\theta)L(0,(\mu_2\theta)^{-1})}\right)\mathcal{O}_\theta.$$

We would like to find finite order anticyclotomic characters θ such that the (algebraic) L-factors in the denominator are p-adic units. We have at our disposal two results on the non-vanishing modulo p of the L-values $L^{\mathrm{alg}}(0,\theta\mu_i^{\pm 1})$ as θ varies in an anticyclotomic $\mathbf{Z}_q$-extension:

Theorem 4.14 (Finis [Fi2] Thm. 1.1). *Let $q \nmid 2\#\mathrm{Cl}(F)$ be a prime split in F, distinct from p. Consider Hecke characters λ of infinity type $\lambda_\infty(z) = z^k\overline{z}^{1-k}$ for a fixed positive integer k with $\lambda^* = \lambda$ (where $\lambda^*(x) = \lambda(\overline{x})^{-1}|x|_{\mathbf{A}_F}$), conductor dividing dd_Fq^∞ for some fixed d, global root number $W(\lambda) = 1$, and such that no inert primes congruent to -1 mod p divide the conductor of λ with multiplicity one. Then for all but finitely many such Hecke characters*

$$L(0,\lambda)W_p(\lambda)\Omega^{1-2k}(k-1)!(\frac{2\pi}{\sqrt{d_F}})^{k-1} \text{ is a } p-\text{adic unit.}$$

The p-adic root number $W_p(\lambda)$ is defined as $p^{-\mathrm{ord}_{v_0}(f_\lambda)} \cdot \tau_{v_0}(\lambda_{v_0})$ for v_0 the place corresponding to $\mathfrak{p}$. For the definition of the Gauss sum τ_{v_0} see Section 2.6.

Hida has announced a similar result (for general CM fields). The pre-print [Hi04b] includes the theorem:

Theorem 4.15 ([Hi04b] Theorem 4.3). *Fix a character λ of split conductor (i.e., such that the conductor is a product of primes split in $F/\mathbf{Q}$) with infinity type $\lambda_\infty(z) = z^k\left(\frac{z}{\overline{z}}\right)^l$ for $k > 0$ and $l \geq 0$. Then*

$$L^{\mathrm{alg}}(\lambda\theta, 0) \text{ is a } p-\text{adic unit}$$

for all but finitely many finite-order anticyclotomic characters θ of q-power conductor for a split prime q distinct from p and coprime to the conductor of λ.

From these two results we deduce the following:

Proposition 4.16. *Let μ_1, μ_2 as above. Assume in addition that either*

(a) the characters satisfy $\mu_i^c = \overline{\mu}_i$ (for this infinity type this coincides with $\mu_i^* = \mu_i$) and that no inert primes congruent to $-1 \mod p$ divide either of the conductors of μ_i with multiplicity one

(b) or the characters μ_i have split conductor.

Then there exists a prime q and a finite order anticyclotomic character θ of q-power conductor such that $L^{\mathrm{alg}}(0, \mu_1\theta)$ and $L^{\mathrm{alg}}(0, (\mu_2\theta)^{-1})$ lie in $\mathcal{O}_\theta^*$.

Proof. All that remains to show is that for (a) $W_p(\lambda)$ is a p-adic unit (we take $k = 1$ in Finis' Theorem). This follows from the definition of the Gauss sum and (the proof of) Lemma 3.21. □

With this we have also proven:

Theorem 4.17. *Under the same assumptions as the previous proposition we have*

$$\delta([\mathrm{Eis}(\Psi_\phi)]) \subseteq L^{\mathrm{alg}}(0, \mu_1/\mu_2)\mathcal{O}_\phi.$$

Theorem 1.2 in the introduction follows from Theorem 4.17:

Corollary 4.18 (Theorem 1.2). *Let χ be a Hecke character of infinity type z^2 such that $\chi^c = \overline{\chi}$. Assume also that no inert primes congruent to $-1 \mod p$ or factors of p divide the conductor of χ. Then there exist characters μ_i such that $\chi = \mu_1/\mu_2$ and $\delta([\mathrm{Eis}(\Psi_{(\mu_1,\mu_2)})]) \subset (L^{\mathrm{alg}}(0, \chi))$.*

Proof. It suffices to show that χ can be factored as μ_1/μ_2 with characters satisfying the conditions of the previous theorem. Put $\mu_1 = \mu_G\chi$ and $\mu_2 = \mu_G$, where μ_G is the character from Lemma 3.18. Since μ_G is ramified only at the ramified places in F and satisfies $\mu_G^c = \overline{\mu}_G$ this choice $\phi = (\mu_1, \mu_2)$ satisfies the condition (a) of Proposition 4.16. □

CHAPTER V

The torsion problem

Recall from Lemma 3.24 that $[\mathrm{res}(\mathrm{Eis}(\Psi_{(\mu_1,\mu_2)}))] \in \tilde{H}^1(\partial\overline{S}_{K_f^s}, \mathcal{O}_\chi)$ if we assume $\chi = \mu_1/\mu_2$ to be anticyclotomic (see Definition 3.22 for $\mathcal{O}_\chi$ and F_χ). In Chapter VI we will show that if we can find $c \in H^1(S_{K_f^s}, \mathcal{O}_\chi)$ with the same restriction to the boundary as our Eisenstein cohomology class $[\mathrm{Eis}(\Psi_{(\mu_1,\mu_2)})] \in H^1(S_{K_f^s}, F_\chi)$ then this implies a congruence between a certain integral multiple of the Eisenstein cohomology class and a cohomology class in $H^1_!(S_{K_f}, \mathcal{O}_\chi)$.

The aim of this chapter is to isolate cases where we can find such a class c, which means ruling out congruences of the Hecke eigenvalues of our Eisenstein class with those of torsion classes in H^2_c. The existence of such torsion classes was shown, for example, in R. Taylor's thesis [T], and in calculations by Feldhusen ([F] p.26). We manage to avoid this "torsion problem" after restricting to constant coefficient systems and unramified χ, and excluding the two fields $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$.

Our strategy is to find an involution on the boundary cohomology such that (for each connected component of $\overline{S}_{K_f^s}$)

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \mathcal{O}_\chi) \overset{\mathrm{res}}{\twoheadrightarrow} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{O}_\chi)^-,$$

where the superscript '-' indicates the -1-eigenspace of this involution. We prove the existence of such an involution for all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$, extending a result of Serre for $\mathrm{SL}_2(\mathcal{O})$. After checking that $[\mathrm{res}(\mathrm{Eis}(\Psi_{(\mu_1,\mu_2)}))]$ lies in this -1-eigenspace, we deduce the existence of the integral lift c.

5.1 Involutions and the image of the restriction map

In this section we work with a general arithmetic subgroup Γ. Assuming that we have an orientation-reversing involution on $\Gamma\backslash\overline{\mathbf{H}}_3$ such that

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \mathcal{O}_\chi) \stackrel{\text{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), \mathcal{O}_\chi)^-$$

we show that the map is, in fact, surjective. The existence of such an involution will be shown for maximal arithmetic subgroups in the following sections.

We first recall:

Theorem 5.1 (Poincaré and Lefschetz duality). *Suppose $\Gamma \subset G(\mathbf{Q})$ is an arithmetic subgroup. Let R be a Dedekind domain in which both the lowest common multiple of the orders of stabilizers $|\Gamma_x|$ as well as the greatest common divisor of the indices of torsion-free subgroups of finite index in Γ are invertible. Then there are perfect pairings*

$$H_c^r(\Gamma\backslash\overline{\mathbf{H}}_3, R) \times H^{3-r}(\Gamma\backslash\overline{\mathbf{H}}_3, R) \to R \;\; \textit{for}\; 0 \le r \le 3$$

and

$$H^r(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \times H^{2-r}(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \to R \;\; \textit{for}\; 0 \le r \le 2.$$

Furthermore, the maps in the exact sequence

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \stackrel{\text{res}}{\longrightarrow} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \stackrel{\partial}{\to} H_c^2(\Gamma\backslash\overline{\mathbf{H}}_3, R)$$

are adjoint, i.e.,

$$\langle \text{res}(x), y\rangle = \langle x, \partial(y)\rangle.$$

Proof. Serre states this in the proof of Lemma 11 in [Se70] for field coefficients, [AS] Lemma 1.4.3 proves the perfectness for fields R and [U95] Theorem 1.6 for Dedekind domains as above. Other references for this Lefschetz or "relative" Poincaré duality for oriented manifolds with boundary are [Ma99] Chapter 21, §4 and [G67] (28.18). We use here that $\overline{\mathbf{H}}_3$ is an oriented manifold with boundary and that Γ acts on it properly discontinuously and without reversing orientation.

For the following, we just want to recall the definition of the pairings: Write $M = \Gamma\backslash\overline{\mathbf{H}}_3$. As explained in [Ma99] there is an unique element (called the R-fundamental class) $z_\Gamma \in H_3(M, \partial M, R)$ such that ∂z_Γ is the fundamental class of $\partial M \in H_2(\partial M, R)$ induced by the R-orientation of M. The pairings are then given by the cup product and evaluation on the respective fundamental classes. □

In particular, one deduces the following lemma:

Lemma 5.2 (Poincaré duality and orientation-reversing involutions). *Suppose Γ and R are as in the theorem and that 2 is invertible in R. Let ι be an orientation-reversing involution on $\Gamma\backslash\overline{\mathbf{H}}_3$. Denoting by a superscript + (resp. −) the +1-(resp. −1-) eigenspaces for the induced involutions on cohomology groups, we have perfect pairings*

$$H^r_c(\Gamma\backslash\overline{\mathbf{H}}_3, R)^{\pm} \times H^{3-r}(\Gamma\backslash\overline{\mathbf{H}}_3, R)^{\mp} \to R \text{ for } 0 \leq r \leq 3$$

and

$$H^r(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\pm} \times H^{2-r}(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\mp} \to R \text{ for } 0 \leq r \leq 2.$$

Proof. That ι reverses the orientation on $\Gamma\backslash\overline{\mathbf{H}}_3$ means exactly that $\iota(z_\Gamma) = -z_\Gamma$ for z_Γ as in the proof of the theorem. This implies that +1- and −1-eigenspaces are "self-orthogonal" under the duality pairing, or maximal isotropic subspaces. Since the perfect pairing for the boundary uses the fundamental class ∂z_Γ the same argument applies to the boundary after checking that the connecting homomorphism ∂ is ι-equivariant. □

Lemma 5.3. *Suppose in addition to the conditions of the previous theorem and lemma that R is a complete discrete valuation ring with finite residue field of characteristic $p > 2$. Suppose that we have an involution ι as in the lemma such that*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \stackrel{\text{res}}{\to} H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^{\epsilon},$$

where $\epsilon = +1$ or -1. Then, in fact, the restriction map is surjective.

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R. Since the cohomology modules are finitely generated (so the Mittag-Leffler condition is satisfied for $\varprojlim H^1(\cdot, R/\mathfrak{m}^r)$), it suffices to prove the surjectivity for each $r \in \mathbf{N}$ of

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R/\mathfrak{m}^r) \twoheadrightarrow H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^\epsilon.$$

For these coefficient systems we are dealing with finite groups and can count the number of elements in the image and the eigenspace of the involution; they turn out to be the same. We observe that $H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r) = H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^+ \oplus H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^-$ and that, by the last lemma,

$$\#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^+ = \#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r)^-.$$

Similarly we deduce from the adjointness of res and ∂ and the perfectness of the pairings that $\mathrm{im}(\mathrm{res})^\perp = \mathrm{im}(\mathrm{res})$ and so

$$\#\mathrm{im}(\mathrm{res}) = \frac{1}{2}\#H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R/\mathfrak{m}^r).$$

□

5.2 The involution for $\mathrm{SL}_2(\mathcal{O})$

We first make the following observation that will simplify the treatment of the cohomology of the boundary components:

Lemma 5.4. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma \subset \mathrm{SL}_2(F)$ an arithmetic subgroup, P a parabolic subgroup of $\mathrm{Res}_{F/\mathbf{Q}}(\mathrm{SL}_{2/F})$ with unipotent radical U_P, and R a ring in which 2 is invertible we have*

$$H^1(\Gamma_P, R) \cong H^1(\Gamma_{U_P}, R),$$

where $\Gamma_P = \Gamma \cap P(\mathbf{Q})$ and $\Gamma_{U_P} = \Gamma \cap U_P(\mathbf{Q})$.

Proof. Serre shows in [Se70] Lemme 7 that $\Gamma_{U_P} \lhd \Gamma_P$ and that the quotient $W_P = \Gamma_P/\Gamma_{U_P}$ can be identified with a subgroup of the roots of unity of F, i.e., of $\{\pm 1\}$ since $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$. We recall his argument here: Suppose $P = B^\eta$ for

$\eta \in G(\mathbf{Q})$ (where $B^\eta(\mathbf{Q}) = \eta^{-1}B(\mathbf{Q})\eta$). The parabolic B^η is the stabilizer of a cusp $D_\eta \in \mathbf{P}^1(F)$, the latter determined by the isomorphism $B(\mathbf{Q})\backslash G(\mathbf{Q}) \cong \mathbf{P}^1(F)$ given by $[\eta] = [\begin{pmatrix} a & b \\ c & d \end{pmatrix}] \mapsto D_\eta := [c : d]$. If $g \in B^\eta(\mathbf{Q})$ denote by $\omega(g)$ the element in F^* such that $g.x = \omega(g)x$ for all $x \in D_\eta$. One gets a short exact sequence

$$1 \to U^\eta(\mathbf{Q}) \to B^\eta(\mathbf{Q}) \xrightarrow{\omega} F^* \to 1.$$

The eigenvalues of any element of an arithmetic subgroup are integral. In particular, if $g \in \Gamma_{B^\eta}$ then $\omega(g) \in \mathcal{O}^*$, and we have the short exact sequence

$$1 \to \Gamma_{U^\eta} \to \Gamma_{B^\eta} \xrightarrow{\omega} \mathcal{O}^* \to 1$$

which proves the claim made at the start.

By the Inflation-Restriction sequence we deduce now that

$$H^1(\Gamma_P, R) \xrightarrow{\sim} H^1(\Gamma_{U_P}, R)^{W_P}$$

since $\#W_B = 2 \in R^*$. Now $W_B \subset \{\pm 1\}$ acts trivially on Γ_U, and therefore $H^1(\Gamma_P, R) \simeq H^1(\Gamma_{U_P}, R)$. □

For a general arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$, the set $\{B^\eta : [\eta] \in B(\mathbf{Q})\backslash G(\mathbf{Q})/\Gamma\}$ is a set of representatives for the Γ-conjugacy classes of Borel subgroups. The group U^η is the unipotent radical of B^η. For $D \in \mathbf{P}^1(F)$ let $\Gamma_D = \Gamma \cap U_D$, where U_D is the unipotent subgroup of $\mathrm{SL}_2(F)$ fixing D. Note that if $D_\eta \in \mathbf{P}^1(F)$ corresponds to $[\eta] \in B(\mathbf{Q})\backslash G(\mathbf{Q})$ under the isomorphism of $B(\mathbf{Q})\backslash G(\mathbf{Q}) \cong \mathbf{P}^1(F)$ given by right action on $[0 : 1] \in \mathbf{P}^1(F)$ (see also Lemma 5.4) we have that $U_{D_\eta} = U^\eta(\mathbf{Q})$ and $\Gamma_{D_\eta} = \Gamma \cap U^\eta(\mathbf{Q}) = \Gamma_{U^\eta}$.

Let $U(\Gamma)$ be the direct sum $\oplus_{[D]\in\mathbf{P}^1(F)/\Gamma}\Gamma_D$. Up to canonical isomorphism this is independent of the choice of representatives $[D] \in \mathbf{P}^1(F)/\Gamma$. The inclusion $\Gamma_D \to \Gamma$ defines a homomorphism $\alpha : U(\Gamma) \to \Gamma^{\mathrm{ab}}$.

Serre studies in [Se70] the kernel of $U(\Gamma) \to \Gamma^{\mathrm{ab}}$. For $\Gamma = \mathrm{SL}_2(\mathcal{O})$ he shows (by choosing an appropriate set of representatives of $\mathbf{P}^1(F)/\mathrm{SL}_2(\mathcal{O} \cong \mathrm{Cl}(F))$ that there is a well-defined action of complex conjugation on $U(\mathrm{SL}_2(\mathcal{O}))$ induced by the complex conjugation action on the matrix entries of $G_\infty = \mathrm{GL}_2(\mathbf{C})$. Denoting by U^+ the set of elements of $U(\mathrm{SL}_2(\mathcal{O}))$ invariant under the involution and by U' the set of elements $u + \overline{u}$ for $u \in U(\mathrm{SL}_2(\mathcal{O}))$, his result is as follows:

Theorem 5.5 (Serre [Se70] Théorème 9). *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ the kernel of the homomorphism $\alpha : U(\mathrm{SL}_2(\mathcal{O})) \to \mathrm{SL}_2(\mathcal{O})^{\mathrm{ab}}$ satisfies the inclusions*

$$6U' \subseteq \ker(\alpha) \subseteq U^+.$$

For our purposes we reinterpret this as follows:

Corollary 5.6. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma = \mathrm{SL}_2(\mathcal{O})$, and R a ring in which 2 and 3 is invertible, the image of the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)$$

is contained in the -1-eigenspace of the involution induced by

$$\iota : \mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0} \to \mathbf{H}_3 : (z, t) \mapsto (\overline{z}, t).$$

Proof. We first note that since $\mathrm{SL}_2(\mathcal{O})$ (in fact, even $\mathrm{GL}_2(F)$) has only 2- and 3-torsion (see Proof of Lemma 1.1 in [F]) we have an isomorphism $H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \cong H^1(\Gamma, R)$ by Proposition 2.5. From Section 2.8 we know that $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$ is homotopy equivalent to

$$\coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma} \Gamma_{B^\eta}\backslash\mathbf{H}_3,$$

where $\Gamma_{B^\eta} = \Gamma \cap B^\eta$. That we have, in fact,

$$H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R) \cong \coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma} H^1(\Gamma_{U^\eta}, R) = H^1(U(\Gamma), R)$$

follows from Lemma 5.4.

The involution ι on $\mathbf{H}_3$ extends canonically to $\overline{\mathbf{H}}_3$. One checks that for $\gamma \in \Gamma$ we have $\iota(\gamma.(z, t)) = \overline{\gamma}.\iota(z, t)$. Since $\overline{\Gamma} = \Gamma$ this implies that ι operates on

$\Gamma\backslash\mathbf{H}_3$ and $\Gamma\backslash\overline{\mathbf{H}}_3$, and hence on $\partial(\Gamma\backslash\overline{\mathbf{H}}_3)$. We note that the involution induced on $\coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma}\Gamma_{B^\eta}\backslash\mathbf{H}_3$ (for a choice of representatives η fixed under complex conjugation) is given by

$$[(z,t)]\in\Gamma_{B^\eta}\backslash\mathbf{H}_3\mapsto[(\overline{z},t)]\in\Gamma_{B^{\overline{\eta}}}\backslash\mathbf{H}_3.$$

We define involutions on the singular cohomology groups

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3,R),H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3),R),\text{ and }\coprod_{[\eta]}H^1(\Gamma_{B^\eta}\backslash\mathbf{H}_3,R)$$

via the involution given on singular cocycles by pullback of ι on the corresponding space. The involution on $H^1(U(\Gamma),R)=\mathrm{Hom}(U(\Gamma),R)$ induced by the complex conjugation action on $U(\Gamma)$ in Serre's theorem is given by $\varphi\mapsto\overline{\varphi}$, where $\overline{\varphi}(u):=\varphi(\overline{u})$.

We claim now that under the isomorphism

$$H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3),R)\cong H^1(U(\Gamma),R)$$

the involutions on both sides correspond. As in Proposition 2.5 we get that the isomorphism

$$H^1(\Gamma_{B^\eta}\backslash\mathbf{H}_3,R)\cong H^1(\Gamma_{U^\eta},R)$$

is given on the level of cocycles by mapping a singular 1-cocycle f to

$$\mathcal{G}_{x_0}(f):\gamma\mapsto f([x_0,\gamma.x_0])$$

for some $x_0\in\mathbf{H}_3$, where $[x_0,\gamma.x_0]$ denotes a 1-cycle with endpoints given by x_0 and $\gamma.x_0\in\mathbf{H}_3$. The map on cohomology classes is independent of the choice of the basepoint x_0 and $[x_0,\gamma.x_0]$. We have now

$$\mathcal{G}_{x_0}(\iota^*(f))=\overline{\mathcal{G}_{\iota(x_0)}(f)}\in\mathrm{Hom}(\Gamma_{U^{\overline{\eta}}},R),$$

so the two involutions do indeed correspond.

We can therefore check that the image of the restriction maps is contained in the -1-eigenspace on the level of group cohomology: The restriction map is given by

$$\mathrm{Hom}(\Gamma^{\mathrm{ab}},R)\to\mathrm{Hom}(U(\Gamma),R):\varphi\mapsto\varphi\circ\alpha.$$

By Serre's theorem $0 = \varphi(\alpha(u\overline{u})) = \varphi((\alpha(u)) + \varphi(\alpha(\overline{u}))$, so $\overline{\varphi \circ \alpha}(u) = \varphi(\alpha(\overline{u})) = -\varphi(\alpha(u))$ for any $u \in U(\Gamma)$. □

In order to apply Lemma 5.3 we still have to check that complex conjugation reverses the orientation of $\overline{\mathbf{H}}_3$, and therefore $\Gamma\backslash\overline{\mathbf{H}}_3$, since Γ (and more generally $\mathrm{SL}_2(\mathbf{C})$) acts without reversing orientation, as we will show below. The orientation of $\mathbf{H}_3$ uniquely determines the orientation of $\overline{\mathbf{H}}_3$. By definition, $\mathbf{H}_3$ being orientable means that one can find a consistent choice of generators of $H_3(\mathbf{H}_3, \mathbf{H}_3 - x, R)$ for $x \in \mathbf{H}_3$ (for the definition of relative homology see Section 3.4). By choosing a coordinate neighborhood of x, i.e., an open neighborhood $U \subset \mathbf{H}_3$ containing x homeomorphic to the open unit ball in $\mathbf{R}^3$, one has isomorphisms (see [Ma99] p. 153)

$$H_3(\mathbf{H}_3, \mathbf{H}_3 - x, R) \cong H_3(U, U - x, R) \cong \tilde{H}_2(U - x, R) \cong \tilde{H}_2(S^2, R),$$

where the '~' denotes reduced cohomology groups and S^2 is the standard 2-sphere. This provides the connection to the "geometric" notion of orientation reversing: Rotations preserve the generator of $\tilde{H}_2(S^2, R)$, reflections in planes act by -1.

The planes in hyperbolic 3-space $\mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0}$ are either Euclidean hemispheres or half-planes which are perpendicular to the boundary $\mathbf{C}$ of $\mathbf{H}_3$ (see [EGM] §I.1.1). Complex conjugation on $\mathbf{H}_3$ is exactly a reflection in one of these half-planes, so is orientation-reversing.

Earlier we also claimed that $\mathrm{SL}_2(\mathbf{C})$ acts on $\mathbf{H}_3$ without reversing orientation. This can easily be seen from the geometric definition of the action of $\mathrm{SL}_2(\mathbf{C})$ via the Poincaré extension of the action on $\mathbf{P}^1(\mathbf{C})$ (see Section 2.3): Since the determinant equals one, the action on $\mathbf{P}^1(\mathbf{C})$ is given by an even number of reflections in lines and circles in $\mathbf{C}$. The action on $\mathbf{H}_3$ is therefore given by an even number of reflections in the corresponding hyperbolic planes.

Applying Lemma 5.3 we have therefore proven:

Corollary 5.7. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma = \mathrm{SL}_2(\mathcal{O})$, and R a complete discrete valuation ring in which 2 and 3 are invertible and with finite residue field of characteristic $p > 2$, the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^-$$

surjects onto the -1-eigenspace of the involution induced by

$$\iota : \mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0} \to \mathbf{H}_3 : (z,t) \mapsto (\overline{z}, t).$$

5.3 The involution for other maximal arithmetic subgroups

Any maximal arithmetic subgroup of $\mathrm{SL}_2(F)$ is conjugate to one of the following groups (see [EGM] Prop. 7.4.5): Let $\mathfrak{b}$ be a fractional ideal and

$$H(\mathfrak{b}) := \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(F) | a, d \in \mathcal{O}, b \in \mathfrak{b}, c \in \mathfrak{b}^{-1} \}.$$

In this section we extend Théorème 9 of [Se70] (Theorem 5.5) to these groups. After we had discovered this generalization we found out that it had already been suggested in [BN], but for our application we need more detail than is provided there.

Remark 5.8. Our choice of embeddings of $\Gamma_{B^\eta}\backslash\mathbf{H}_3$ into the adelic boundary $\partial\tilde{S}_{K_f} = B(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty$ in Section 3.3.3 means that the arithmetic subgroups Γ act from the right on the set of cusps $\mathbf{P}^1(F) \cong B(\mathbf{Q})\backslash G(\mathbf{Q})$. All the actions of $H(\mathfrak{b})$ in this section will therefore be written as right actions.

Note that since $H(\mathfrak{b})$ is the stabilizer of any lattice $\mathfrak{m} \oplus \mathfrak{n}$ with $\mathfrak{m}$ and $\mathfrak{n}$ fractional ideals of F such that $\mathfrak{m}^{-1}\mathfrak{n} = \mathfrak{b}$, one can deduce

Lemma 5.9. *Let $\mathfrak{a}, \mathfrak{b}$ be two fractional ideals of F. If $[\mathfrak{a}] = [\mathfrak{b}]$ in $\mathrm{Cl}(F)/\mathrm{Cl}(F)^2$, then $H(\mathfrak{a}) = H(\mathfrak{b})^\gamma$ with $\gamma \in \mathrm{GL}_2(F)$. If the fractional ideals differ by the square of an $\mathcal{O}$-ideal, then γ can be taken to be in $\mathrm{SL}_2(F)$.*

We first generalize a Theorem of Bianchi for $\mathrm{SL}_2(\mathcal{O})$ (see [EGM] Theorem VII 2.4) to $H(\mathfrak{b})$. For this we need the following lemma.

Lemma 5.10. *Let (x_1, x_2), $(y_1, y_2) \in F \times F$. The following are equivalent:*

(1) $x_1\mathfrak{b} + x_2\mathcal{O} = y_1\mathfrak{b} + y_2\mathcal{O}$.

(2) *There exists $\sigma \in H(\mathfrak{b})$ such that $(x_1, x_2) = (y_1, y_2)\sigma$.*

Proof. We follow exactly the proof for $\mathrm{SL}_2(\mathcal{O})$ in [EGM].

(2) $\Rightarrow$ (1) is clear.

(1) $\Rightarrow$ (2): Put $\mathfrak{a} = x_1\mathfrak{b} + x_2\mathcal{O} = y_1\mathfrak{b} + y_2\mathcal{O}$. If $\mathfrak{a} = (0)$, then there is nothing to prove. Otherwise choose an $n \in \mathbf{N}$ and $\theta \in F^*$ such that $\mathfrak{a}^n = (\theta)$. Note that $x_1, y_1 \in \mathfrak{a}\mathfrak{b}^{-1}$. The equations $(x_1\mathfrak{b} + x_2\mathcal{O})\mathfrak{a}^{n-1} = (\theta)$ and $(y_1\mathfrak{b} + y_2\mathcal{O})\mathfrak{a}^{n-1} = (\theta)$ show that there are $\alpha_1, \beta_1 \in \mathfrak{a}^{n-1}\mathfrak{b}$ and $\alpha_2, \beta_2 \in \mathfrak{a}^{n-1}$ with $\theta = \alpha_1 x_1 + \alpha_2 x_2$ and $\theta = \beta_1 y_1 + \beta_2 y_2$.

Put

$$\sigma = \begin{pmatrix} \frac{y_1\alpha_1 + x_2\beta_2}{\theta} & \frac{y_2\alpha_1 - x_2\beta_1}{\theta} \\ \frac{y_1\alpha_2 - x_1\beta_2}{\theta} & \frac{y_2\alpha_2 + x_1\beta_1}{\theta} \end{pmatrix}.$$

It is easy to check that σ lies in $H(\mathfrak{b})$ and satisfies (2). □

Definition 5.11. Define $j : \mathbf{P}^1(F) \to \mathrm{Cl}(F)$ to be the map

$$j([z_1 : z_2]) = [z_1\mathfrak{b} + z_2\mathcal{O}].$$

Clearly, j is well-defined. The preceding lemma now implies

Theorem 5.12. *For $\Gamma = H(\mathfrak{b})$, the induced map*

$$j : \mathbf{P}^1(F)/\Gamma \to \mathrm{Cl}(F)$$

is a bijection.

Proof. In light of lemma 5.10 the only thing left to show is the surjectivity of j. Given a class in $\mathrm{Cl}(F)$ take $\mathfrak{a} \subset \mathcal{O}$ representing it. By the Chinese Remainder Theorem one can choose $z_2 \in \mathcal{O}$ such that

- $\mathrm{ord}_{\wp}(z_2) = \mathrm{ord}_{\wp}(\mathfrak{a})$ if $\wp | \mathfrak{a}$.

- $\mathrm{ord}_\wp(z_2) = 0$ if $\wp \nmid \mathfrak{a}, \mathrm{ord}_\wp(\mathfrak{b}) \neq 0$.

Then one chooses z_1 such that

- $\mathrm{ord}_\wp(z_1\mathfrak{b}) > \mathrm{ord}_\wp(z_2)$ if $\wp|\mathfrak{a}$ or $\mathrm{ord}_\wp(\mathfrak{b}) \neq 0$.
- $\mathrm{ord}_\wp(z_1\mathfrak{b}) = 0$ if $\wp|z_2, \wp \nmid \mathfrak{a}$, and $\mathrm{ord}_\wp(\mathfrak{b}) = 0$.

These choices ensure that $\mathrm{ord}_\wp(z_1\mathfrak{b} + z_2\mathcal{O}) = \mathrm{ord}_\wp(\mathfrak{a})$ for all prime ideals $\wp$.

□

Recall that for $D \in \mathbf{P}^1(F)$ we put $\Gamma_D = \Gamma \cap U_D$, where U_D is the unipotent subgroup of $\mathrm{SL}_2(F)$ fixing D. The Theorem implies

Corollary 5.13. *For $\Gamma = H(\mathfrak{b})$, if $j([x_1 : x_2]) = j([y_1 : y_2])$ then $\Gamma_{[x_1:x_2]}$ is conjugate in Γ to $\Gamma_{[y_1:y_2]}$.*

5.3.1 Representing elements of $\Gamma_{[z_1:z_2]}$

Lemma 5.14. *For any fractional ideal $\mathfrak{a}$ (or projective $\mathcal{O}$-module of rank 1),*

$$\Lambda^2(\mathfrak{a}) = 0.$$

Proof. One has $\mathfrak{a} \oplus \mathfrak{a}^{-1} \cong \mathcal{O} \oplus \mathcal{O}$ (see, for example [Bour] VII, §4, Prop. 24). This implies $\Lambda^2(\mathfrak{a} \oplus \mathfrak{a}^{-1}) \cong \Lambda^2(\mathcal{O}^2) = \mathcal{O}$. Since $\Lambda^2(\mathfrak{a} \oplus \mathfrak{a}^{-1}) = \mathfrak{a} \otimes \mathfrak{a}^{-1} \oplus \Lambda^2(\mathfrak{a}) \oplus \Lambda^2(\mathfrak{a}^{-1})$ we deduce $\Lambda^2(\mathfrak{a}) = 0$.

Alternatively, observe that the localization of $\mathfrak{a}$ at any prime ideal $\wp$ of $\mathcal{O}$ is a free $\mathcal{O}_\wp$-module of rank 1. Since the exterior product commutes with localization this also proves the Lemma. □

Recall the definition of $\Gamma_{[z_1:z_2]}$ from the start of Section 5.2. The following Lemma will be useful for studying the kernel of $\alpha : U(\Gamma) = \oplus_{[D]\in\mathbf{P}^1(F)/\Gamma}\Gamma_D \to \Gamma^{\mathrm{ab}}$:

Lemma 5.15. *For $\Gamma = H(\mathfrak{b})$, $\Gamma_{[z_1:z_2]}$ is conjugate in $H(\mathfrak{b})$ to*

$$\{\theta \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \theta^{-1} : t \in \mathfrak{a}^{-2}\mathfrak{b}\},$$

where $\mathfrak{a} = z_1\mathfrak{b} + z_2\mathcal{O}$ *and* θ *is an isomorphism* $\mathcal{O} \oplus \mathfrak{b} \xrightarrow{\sim} \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ *of determinant 1, i.e., such that its second exterior power*

$$\Lambda^2\theta : \Lambda^2(\mathcal{O} \oplus \mathfrak{b}) = \mathfrak{b} \to \Lambda^2(\mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}) = \mathfrak{a} \otimes \mathfrak{a}^{-1}\mathfrak{b} = \mathfrak{b}$$

is the identity.

Proof. The main change to Serre's method in [Se70] §3.6 is that we consider the lattice $L := \mathcal{O} \oplus \mathfrak{b}$ instead of $\mathcal{O}^2$. We claim there exists a projective rank 1 submodule E of L containing a multiple of (z_1, z_2). Let E be the kernel of the $\mathcal{O}$-homomorphism $L = \mathcal{O} \oplus \mathfrak{b} \to F$ given by $(x, y) \mapsto yz_1 - xz_2$. Since the image is $\mathfrak{a} = z_1\mathfrak{b} + z_2\mathcal{O}$, we get $L/E \cong \mathfrak{a}$, so L/E is projective of rank 1 and L decomposes as $E \oplus L/E$.

By definition $\Gamma_{[z_1:z_2]}$ fixes $L \cap \{\lambda(z_1, z_2), \lambda \in F\}$, but this is exactly E. Since $\Gamma_{[z_1:z_2]}$ is unipotent it can therefore be identified with $\mathrm{Hom}_{\mathcal{O}}(L/E, E)$. Using the exterior product $\mathfrak{b} = \Lambda^2(L) = \Lambda^2(E \oplus L/E) = E \otimes_{\mathcal{O}} L/E$, we get that E is isomorphic to $(L/E)^{-1} \otimes \mathfrak{b}$ (here we use the preceding lemma). This implies an isomorphism $\mathrm{Hom}_{\mathcal{O}}(L/E, E) = (L/E)^{-1} \otimes E \cong (L/E)^{-1} \otimes (L/E)^{-1} \otimes \mathfrak{b} \cong \mathfrak{a}^{-2}\mathfrak{b}$. Choosing an isomorphism $\theta : L \to L/E \oplus E \cong \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1 we can represent $\Gamma_{[z_1:z_2]}$ as stated above. □

Remark 5.16. 1. Alternatively, $\Gamma_{[z_1:z_2]}$ is conjugate to $\{\theta' \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \theta'^{-1} : t \in \mathfrak{a}^{-2}\mathfrak{b}\}$ for an isomorphism $\theta' : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a}^{-1}\mathfrak{b} \oplus \mathfrak{a}$ of determinant 1. Up to conjugation by an element in $H(\mathfrak{b})$, θ' is given by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \theta$.

2. For $\Gamma = \mathrm{SL}_2(\mathcal{O})$ our $\Gamma_{[z_1:z_2]}$ equals $\Gamma_{[\mathfrak{a}^{-1}\mathfrak{b}]}$ in Serre's notation in [Se70] §3.6, not $\Gamma_{[\mathfrak{a}]}$. This follows from our different choice of the isomorphism $j : \mathbf{P}^1(F)/\Gamma \to \mathrm{Cl}(F)$.

5.3.2 The involution on $U(\Gamma)$

If the class of $\mathfrak{b}$ in $\mathrm{Cl}(F)$ is a square, $H(\mathfrak{b})$ is isomorphic to $\mathrm{SL}_2(\mathcal{O})$ by Lemma 5.9, and the involution on $U(\mathrm{SL}_2(\mathcal{O}))$ induced by complex conjugation and Serre's Théorème 9 can easily be transferred to $U(H(\mathfrak{b}))$. We therefore turn our attention to the case when $[\mathfrak{b}]$ is not a square in $\mathrm{Cl}(F)$. Note that this implies that $[\mathfrak{b}]$ has even order, since any odd order class can be written as a square.

Definition 5.17. Define an involution on $H(\mathfrak{b})$ to be the composition of complex conjugation with an Atkin-Lehner involution, i.e. by

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto A\overline{H}A^{-1} = \begin{pmatrix} \overline{d} & -\mathrm{Nm}(\mathfrak{b})\overline{c} \\ -\overline{b}\mathrm{Nm}(\mathfrak{b})^{-1} & \overline{a} \end{pmatrix},$$

where $A = \begin{pmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{pmatrix}$.

Like Serre, we will choose a set of representatives for the cusps $\mathbf{P}^1(F)/H(\mathfrak{b})$ on which this involution acts. For this we observe that if $\Gamma_{[z_1:z_2]}$ fixes $[z_1 : z_2]$ then $A\overline{\Gamma}_{[z_1:z_2]}A^{-1}$ fixes $[\overline{z}_1 : \overline{z}_2]A^{-1} = [\overline{z}_2 : -\mathrm{Nm}(\mathfrak{b})\overline{z}_1]$. We use the isomorphism $j : \mathbf{P}^1(F)/H(\mathfrak{b}) \to \mathrm{Cl}(F)$ to show that this action on the cusps is fixpoint-free. We observe that if $j([z_1 : z_2]) = \mathfrak{a}$ then $j([\overline{z}_1 : \overline{z}_2]A^{-1}) = [\overline{z}_2\mathfrak{b} + \mathrm{Nm}(\mathfrak{b})\overline{z}_1\mathcal{O}] = [\overline{\mathfrak{a}}\mathfrak{b}]$. Note that $[\mathfrak{a}] \neq [\overline{\mathfrak{a}}\mathfrak{b}]$ in $\mathrm{Cl}(F)$ since otherwise $[\mathfrak{a}^2] = [\mathrm{Nm}(\mathfrak{a})\mathfrak{b}] = [\mathfrak{b}]$, i.e., $[\mathfrak{b}]$ a square, contradicting our hypothesis. So $\mathrm{Cl}(F)$ can be partitioned into pairs $(\mathfrak{a}_i, \overline{\mathfrak{a}_i}\mathfrak{b})$.

Choosing $[z_1^i : z_2^i] \in \mathbf{P}^1(F)$ such that $\mathfrak{a}_i = z_1^i\mathfrak{b} + z_2^i\mathcal{O}$ we obtain

$$U(H(\mathfrak{b})) = \bigoplus_{(\mathfrak{a}_i, \overline{\mathfrak{a}_i}\mathfrak{b})} (\Gamma_{[z_1^i:z_2^i]} \oplus A\overline{\Gamma}_{[z_1^i:z_2^i]}A^{-1}).$$

Our choice of representatives of $\mathbf{P}^1(F)/H(\mathfrak{b})$ shows that the involution operates on $U(H(\mathfrak{b}))$ and, in fact, by identifying $\Gamma_{[z_1^i:z_2^i]}$ with $\{\theta \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \theta^{-1} : s \in \mathfrak{a}_i^{-2}\mathfrak{b}\}$ for $\theta : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a}_i \oplus \mathfrak{a}_i^{-1}\mathfrak{b}$ and $A\overline{\Gamma}_{[z_1^i:z_2^i]}A^{-1}$ with $\{\theta' \begin{pmatrix} 1 & 0 \\ -t & 1 \end{pmatrix} \theta'^{-1} : t \in \overline{\mathfrak{a}_i}^{-2}\mathfrak{b}^{-1}\}$ for $\theta' = A\overline{\theta}A^{-1} : \mathcal{O} \oplus \mathfrak{b} \to \overline{\mathfrak{a}_i}^{-1} \oplus \overline{\mathfrak{a}_i}\mathfrak{b}$, we can describe the involution on each of the pairs as

$$(s, t) \in \mathfrak{a}_i^{-2}\mathfrak{b} \oplus \overline{\mathfrak{a}_i}^{-2}\mathfrak{b}^{-1} \mapsto (\overline{t}\mathrm{Nm}(\mathfrak{b}), \overline{s}\mathrm{Nm}(\mathfrak{b})^{-1}).$$

5.3.3 Generalization of Serre's Théorème 9

Denote by U^+ the set of elements of $U(H(\mathfrak{b}))$ invariant under the involution $H \mapsto A\overline{H}A^{-1}$, and by U' the set of elements $u + A\overline{u}A^{-1}$ for $u \in U(H(\mathfrak{b}))$.

Theorem 5.18. *For $\Gamma = H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, the kernel N of the homomorphism*

$$\alpha : U(\Gamma) \to \Gamma^{\mathrm{ab}}$$

coming from the inclusion $\Gamma_D \hookrightarrow \Gamma$ for $D \in \mathbf{P}^1(F)$ satisfies $6U' \subset N \subset U^+$.

Remark 5.19. Given Lemma 5.9, this provides the extension of Serre's Theorem to all maximal arithmetic subgroups of $\mathrm{SL}_2(F)$.

Proof. With small modifications, we follow Serre's proof of his Théorème 9. As in Serre's case, it suffices to prove the inclusion $6U' \subset N$, i.e. that $6(u + A\overline{u}A^{-1})$ maps to something in the commutator $[H(\mathfrak{b}), H(\mathfrak{b})]$:

Suppose that we have $6U' \subset N$, but that there exists an element $u \in N$ not contained in U^+. Then the subgroup of N generated by $6U'$ and u has rank $\#\mathrm{Cl}(F)+1$. This contradicts the fact that the kernel of α has rank $\#\mathrm{Cl}(F)$ (see [Se70] Théorème 7). (The latter is proven by showing dually that the rank of the image of the restriction map $H^1(H(\mathfrak{b})\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(H(\mathfrak{b})\backslash\overline{\mathbf{H}}_3), R)$ has half the rank of that of the boundary cohomology. This we showed in the proof of Lemma 5.3).

To prove $6U' \subset N$ now we make use of Serre's Proposition 6:

Proposition 5.20 ([Se70] Proposition 6). *Let $\mathfrak{q}$ be a fractional ideal of F and let $t \in \mathfrak{q}$ and $t' = \overline{t}/\mathrm{Nm}(\mathfrak{q})$ so that $t' \in \mathfrak{q}^{-1}$. Put $x_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $y_t = \begin{pmatrix} 1 & 0 \\ -t' & 1 \end{pmatrix}$. Then $(x_t y_t)^6$ lies in the commutator subgroup of $H(\mathfrak{q})$.*

Put $\mathfrak{a} := z_1\mathfrak{b} + z_2\mathcal{O}$. If $u \in \Gamma_{[z_1:z_2]}$, identify it with $\theta^{-1}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\theta$ for some $t \in \mathfrak{a}^{-2}\mathfrak{b}$ and $\theta : \mathcal{O} \oplus \mathfrak{b} \to \mathfrak{a} \oplus \mathfrak{a}^{-1}\mathfrak{b}$ of determinant 1. One easily checks that $A\overline{u}A^{-1}$ then corresponds to $(A\overline{\theta}A^{-1})\begin{pmatrix} 1 & 0 \\ -\overline{t}\mathrm{Nm}(\mathfrak{b})^{-1} & 1 \end{pmatrix}(A\overline{\theta}A^{-1})$. Like Serre, we use now that since $[\overline{\mathfrak{a}}] = [\mathfrak{a}^{-1}]$, $A\overline{u}A^{-1}$ is also given by Corollary 5.13 by $B^{-1}\theta^{-1}\begin{pmatrix} 1 & 0 \\ -t' & 1 \end{pmatrix}\theta B$ for $t' = \overline{t}\mathrm{Nm}(\mathfrak{b})^{-1}\mathrm{Nm}(\mathfrak{a})^2$ and $B \in H(\mathfrak{b})$ taking $\begin{pmatrix} \mathrm{Nm}(\mathfrak{b})\overline{z}_2 \\ \overline{z}_1 \end{pmatrix}$ to $\mathrm{Nm}(\mathfrak{a})^{-1}\begin{pmatrix} \mathrm{Nm}(\mathfrak{b})\overline{z}_2 \\ \overline{z}_1 \end{pmatrix}$.

Since $\theta^{-1}x_t y_t \theta$ is a representative of $u + BA\overline{u}A^{-1}B^{-1}$, we deduce from the above Proposition with $\mathfrak{q} = \mathfrak{a}^{-2}\mathfrak{b}$ that $6(u + BA\overline{u}A^{-1}B^{-1})$ and therefore $6(u + A\overline{u}A^{-1})$ lie in $[H(\mathfrak{b}), H(\mathfrak{b})]$. □

Remark 5.21. Since $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O})$ it is possible to unify the treatment of all maximal arithmetic subgroups $H(\mathfrak{b})$ independent of $\mathfrak{b}$ being a square in the class group or not. However, since it would not significantly simplify notation or exposition we did not pursue this here.

We again want to reformulate our result in the following form:

Corollary 5.22. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma = H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, and R a ring in which 2 and 3 is invertible, the image of the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)$$

is contained in the -1-eigenspace of the involution induced by

$$\iota : \mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0} \to \mathbf{H}_3 : (z, t) \mapsto A.(\overline{z}, t)$$

for $A = \begin{pmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{pmatrix}$.

Proof. It is easy to check that $A\overline{\Gamma}A^{-1} = \Gamma$ and that

$$\iota(\gamma.(z, t)) = (A\overline{\gamma}A^{-1}).\iota(z, t).$$

Now we proceed exactly as in the proof of Corollary 5.6. □

To be able to apply Lemma 5.3 we again show that this involution is orientation-reversing on $\mathbf{H}_3$. Since $A \in \mathrm{GL}_2(\mathbf{C})$ acts on $\mathbf{H}_3$ via $A' = (\det(A)^{-\frac{1}{2}})A \in \mathrm{SL}_2(\mathbf{C})$, its action on $\mathbf{H}_3$ preserves the orientation (see end of Section 5.2, before Corollary 5.7). Recalling further that the action of complex conjugation is orientation-reversing, our involution is therefore orientation-reversing. For future reference we record the application of Lemma 5.3 to our involution:

Corollary 5.23. *For imaginary quadratic fields F other than $\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$, $\Gamma = H(\mathfrak{b})$ with $[\mathfrak{b}]$ a non-square in $\mathrm{Cl}(F)$, and R a complete discrete valuation ring in which 2 and 3 are invertible and with finite residue field of characteristic $p > 2$, the restriction map*

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, R) \to H^1(\partial(\Gamma\backslash\overline{\mathbf{H}}_3), R)^-$$

surjects onto the -1-eigenspace of the involution induced by

$$\iota : \mathbf{H}_3 = \mathbf{C} \times \mathbf{R}_{>0} \to \mathbf{H}_3 : (z,t) \mapsto A.(\overline{z}, t)$$

for $A = \begin{pmatrix} 0 & 1 \\ -\mathrm{Nm}(\mathfrak{b})^{-1} & 0 \end{pmatrix}$.

5.4 Unramified characters χ

In Chapter III we defined an Eisenstein cohomology class associated to (μ_1, μ_2) on an adelic symmetric space $S_{K_f^s}$ for a specific choice of K_f^s. In this section we will show that for unramified characters $\chi = \mu_1/\mu_2$ we can always write the corresponding $S_{K_f^s}$ as a disjoint union of $\Gamma\backslash\mathbf{H}_3$ with $\Gamma = H(\mathfrak{b})$ for fractional ideals $\mathfrak{b}$. This allows us to apply our results for maximal arithmetic subgroups from the previous sections by considering the restriction maps to the boundary separately for each connected component.

We recall now from Section 2.3: The space S_{K_f} has several connected components, in fact, strong approximation implies that the fibers of the determinant map

$$S_{K_f} = G(\mathbf{Q})\backslash G(\mathbf{A})/(K_f K_\infty) \twoheadrightarrow H_K := \mathbf{A}_F^*/\det(K)F^*$$

(where $K = K_f K_\infty$) are connected. Any $\xi \in G(\mathbf{A}_f)$ gives rise to an injection $j_\xi : G_\infty \to G(\mathbf{A})$ with $j_\xi(g_\infty) = (g_\infty, \xi)$ and, after taking quotients, to a component

$$\Gamma_\xi\backslash G_\infty/K_\infty \to G(\mathbf{Q})\backslash G(\mathbf{A})/K,$$

where $\Gamma_\xi := G(\mathbf{Q}) \cap \xi K_f \xi^{-1}$. This component is the fiber over $\det(\xi)$.

Let F now be any imaginary quadratic field different from $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$. Let $\chi : F^*\backslash\mathbf{A}_F^*/\prod_v \mathcal{O}_v^* \to \mathbf{C}^*$ be an unramified Hecke character of infinity type $\chi_\infty(z) = z^2$. We consider $(\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ with μ_1, μ_2 Hecke characters of infinity type z and z^{-1}, respectively, such that $\chi = \mu_1/\mu_2$. Let

$$K_f = K_f^s = \prod_{v \in S} U^1(\mathfrak{M}_{1,v}) \prod_{v \notin S} \mathrm{GL}_2(\mathcal{O}_v),$$

for S the set of places where either of the μ_i are ramified (they have to be ramified because their infinity types are z or z^{-1}). Here $U^1(\mathfrak{M}_{1,v}) = \{k \in \mathrm{GL}_2(\mathcal{O}_v) : \det(k) \equiv 1 \mod \mathfrak{M}_{1,v}\}$ and $\mathfrak{M}_{1,v}$ is the conductor of $\mu_{1,v}$. Put $K^s = K_f^s K_\infty$.

Assumption 5.24. *The only unit in $\mathcal{O}^*$ congruent to 1 modulo $\mathfrak{M}_{1,v}$ for some $v \in S$ is 1.*

Under this assumption we have $K_f^s \cap \mathrm{GL}_2(F) = SL_2(\mathcal{O})$.

We want to find a set $\{t_i \in G(\mathbf{A}_f), i = 1 \dots h_{K^s}\}$ with $\{\det(t_i)\}$ providing a system of representatives for H_{K^s} such that the Γ_{t_i} equal $H(\mathfrak{b}_i)$ for appropriate fractional ideals. For a finite idele a, denote by (a) the corresponding fractional ideal. Since in our case $\det(K_f^s)$ has finite index in $\prod_v \mathcal{O}_v^*$ (H_{K^s} is a generalized ray class group modulo the conductor of μ_1), $\{(\det(t_i))\}$ has to run through a number of copies of a set of representatives for the ideal class group $\mathrm{Cl}(F)$.

We first consider the special case where $\#H_{K^s}$ is odd (this requires the class number of F to be odd, but also imposes restrictions on K_f^s and the ramification of our factorization $\chi = \mu_1/\mu_2$): If H_{K^s} is a group of odd order h_{K^s} then any element of H_{K^s} is a square. Rather than the usual choice of t_i as $\begin{pmatrix} a_i & 0 \\ 0 & 1 \end{pmatrix}$ such that $\{a_i \in \mathbf{A}_{F,f}^*, i = 1 \dots h_{K^s}\}$ is a set of representatives for H_{K^s}, we can therefore take $t_i = \begin{pmatrix} b_i & 0 \\ 0 & b_i \end{pmatrix} \in G(\mathbf{A}_f)$, where $[b_i]^2 = [a_i] \in H_{K^s}$. This means that in this case we can ensure that all Γ_{t_i} equal $\Gamma := G(\mathbf{Q}) \cap K_f^s = \mathrm{SL}_2(\mathcal{O})$.

Next we consider the case where $\#H_{K^s}$ is even. First, we choose a system of representatives $\{\gamma_j\}$ of

$$\ker(H_{K^s} \to \mathrm{Cl}(F)) \cong \mathcal{O}^* \backslash \prod_v \mathcal{O}_v^* / \det(K_f^s).$$

Then take a set of representatives $\{a_k\}$ of $\mathrm{Cl}(F)/(\mathrm{Cl}(F))^2$ in $\mathbf{A}_{F,f}^*$ (represent the principal ideals by (1)). Lastly, we choose a set $\{b_m^2\}$ representing $\mathrm{Cl}(F)^2$.

Now we can define $\{t_i\}$ as follows: The set is given by the elements

$$\begin{pmatrix} \gamma_j a_k b_m & 0 \\ 0 & b_m \end{pmatrix} \in G(\mathbf{A}_f),$$

as j, k and m run through their indexing sets.

We obtain a decomposition

$$S_{K_f^s} \cong \coprod_{i=1}^{h_{K^s}} \Gamma_{t_i} \backslash G_\infty / K_\infty \cong \coprod_{i=1}^{h_{K^s}} \Gamma_{t_i} \backslash \mathbf{H}_3,$$

where $\Gamma_{t_i} = G(\mathbf{Q}) \cap t_i K_f^s t_i^{-1}$, with the i-th connected component $\Gamma_{t_i} \backslash \mathbf{H}_3$ being embedded via j_{t_i}.

Note that if $t_i = \begin{pmatrix} \gamma_j a_k b_m & 0 \\ 0 & b_m \end{pmatrix}$, the associated $\Gamma_{t_i} = H((a_k))$ under Assumption 5.24. Also, by construction, either $a_k = 1$ or $[(a_k)]$ is not a square in $\mathrm{Cl}(F)$.

5.5 Integral lift of constant term

In this section we show that for $\chi = \mu_1/\mu_2$ an unramified character we can lift the constant term of the Eisenstein cohomology class to an integral class, i.e., that there exists $c \in \tilde{H}^1(S_{K_f^s}, \mathcal{O}_\chi)$ with the same restriction to the boundary as the Eisenstein cohomology class.

Our setup is as follows: Let F be an imaginary quadratic field, distinct from $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$. Suppose $\mathfrak{p}$ is a prime of F such that the underlying rational prime p is greater than 3 and splits in F.

Let $\mu_1, \mu_2 : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ be Hecke characters of infinity type z and z^{-1}, respectively, such that $\chi = \mu_1/\mu_2$ is unramified. Denote by S the set of places where the μ_i are ramified. Assume in addition that 1 is the only unit in $\mathcal{O}^*$ congruent to 1 modulo $\mathfrak{M}_{1,v}$ for some $v \in S$ (i.e., that Assumption 5.24 holds). Put $\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$. Let $\mathcal{O}_\chi$ denote the ring of integers in the finite extension F_χ of $F_\mathfrak{p}$ obtained by adjoining the values of both $\mu_{i,f}$ and $L^{\text{alg}}(0, \chi)$.

In the last section we showed that under these conditions there exist $\{t_i \in G(\mathbf{A}_f)\}$ such that $\{\det(t_i)\}$ is a system of representatives for H_{K^s} and such that the Γ_{t_i} are all either $\mathrm{SL}_2(\mathcal{O})$ or $H(\mathfrak{b})$ for fractional ideals $\mathfrak{b}$ whose classes are non-square in $\mathrm{Cl}(F)$. Since unramified characters are anticyclotomic (see Lemma 3.16) Lemma 3.24 applies so that $[\mathrm{res}(\mathrm{Eis}(\omega_0(\phi, \Psi_\phi)))]$ is integral. We can further show:

Proposition 5.25.

$$[\mathrm{res}(\mathrm{Eis}(\Psi_\phi))] \in \widetilde{H}^1(\partial \tilde{S}_{K_f^s}, \mathcal{O}_\chi)^-,$$

where the latter is defined via the isomorphism to

$$\bigoplus_{[\det(t_i)] \in H_{K^s}} \widetilde{H}^1(\partial(\Gamma_{t_i}\backslash\overline{\mathbf{H}}_3), \mathcal{O}_\chi)^-$$

for the choice of t_i's from the last section and where the involutions on each of the connected components are defined as in Corollaries 5.7 (if $\Gamma_{t_i} = \mathrm{SL}_2(\mathcal{O})$) and 5.23 (if $\Gamma_{t_i} = H(\mathfrak{b})$ for $\mathfrak{b}$ a non-square in $\mathrm{Cl}(F)$).

Remark 5.26. Together with Corollaries 5.7 and 5.23 this shows the existence of an integral lift of the constant term. Note that $\mathcal{O}_\chi$ is the ring of integers in a finite extension of $F_\mathfrak{p}$ so that the conditions in these corollaries are satisfied.

Proof. We shorten the notation to

$$\omega_0(\Psi_\phi)) := \omega_0(\phi, \Psi_\phi)) \text{ and } \omega_0(\Psi_{w_0.\phi}) := \omega_0(w_0.\phi, \Psi_{w_0.\phi}).$$

Under the assumptions in this section we have that

$$[\mathrm{res}(\mathrm{Eis}(\omega_0(\Psi_\phi)))] = [\omega_0(\Psi_\phi) - \omega_0(\Psi_{w_0.\phi})] \in \widetilde{H}^1(\partial \tilde{S}_{K_f^s}, \mathcal{O}_\chi)$$

(see the proof of Lemma 3.24). Recall that the boundary of the Borel-Serre compactification of the adelic symmetric space is homotopy equivalent to

$$\partial \tilde{S}^G_{K_f^s} = B(\mathbf{Q})\backslash G(\mathbf{A})/K_f^s K_\infty \cong \bigoplus_{[\det(t_i)] \in H_{K^s}} \bigoplus_{[\eta] \in \mathbf{P}^1(F)/\Gamma_{t_i}} \Gamma_{t_i, B^\eta}\backslash \mathbf{H}_3,$$

where $\Gamma_{t_i,B^\eta} = \Gamma_{t_i} \cap \eta^{-1}B(\mathbf{Q})\eta$ for $\eta \in G(\mathbf{Q})$. We will consider the restriction maps to the boundary separately for each connected component $\Gamma_{t_i}\backslash\mathbf{H}_3$ and use the isomorphisms with group cohomology:

$$\widetilde{H}^1(\Gamma_{t_i}\backslash\mathbf{H}_3,\mathcal{O}_\chi) \cong \widetilde{H}^1(\Gamma_{t_i},\mathcal{O}_\chi) \stackrel{\text{res}}{\to} \widetilde{H}^1(\partial(\Gamma_{t_i}\backslash\overline{\mathbf{H}}_3),\mathcal{O}_\chi) \cong \bigoplus_{[\eta]\in\mathbf{P}^1(F)/\Gamma_{t_i}} \widetilde{H}^1(\Gamma_{t_i,B^\eta},\mathcal{O}_\chi).$$

We need to show therefore that the cohomology class determined by the constant term lies in the -1-eigenspace of the involution induced by $u \mapsto \overline{u}$ for $\Gamma_{t_i} = \mathrm{SL}_2(\mathcal{O})$ and by $u \mapsto A\overline{u}A^{-1}$ for $\Gamma_{t_i} = H(\mathfrak{b})$.

The restriction to $\Gamma_{t_i,P}\backslash\mathbf{H}_3$ was denoted by $[\mathrm{res}_P^{t_i}(\mathrm{Eis}(\omega_0(\Psi_\phi)))]$ and equals, by Lemma 3.6 and the proof of Lemma 3.24, the class of

$$\omega_0(\Psi_\phi)_P^{t_i} - \omega_0(\Psi_{w_0.\phi})_P^{t_i}.$$

For $P = B^\eta$ the image of the latter under $\mathcal{G}_{\eta_\infty^{-1}K_\infty}$ (we will drop the subscript from now on) was calculated in Lemma 3.10 and is given in our case (when $m = n = 0$) by:

$$\begin{aligned} \mathcal{G}(\omega_0(\Psi_\phi)_{B^\eta}^{t_i})(\eta_\infty^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\eta_\infty) &= x\Psi_\phi(\eta_f t_i)\\ \mathcal{G}(\omega_0(\Psi_{w_0.\phi})_{B^\eta}^{t_i})(\eta_\infty^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\eta_\infty) &= \overline{x}\Psi_{w_0.\phi}(\eta_f t_i). \end{aligned}$$

Note that by Lemma 5.4 we can restrict to U^η.

Case (1). We again first treat the notationally simpler case where all Γ_{t_i} equal $\mathrm{SL}_2(\mathcal{O})$ and the involution is complex conjugation on the matrix entries.

We claim that

$$\mathcal{G}(\omega_0(\Psi_\phi)_{B^\eta}^{t_i})(g) = \mathcal{G}(\omega_0(\Psi_{w_0.\phi})_{B^{\overline{\eta}}}^{t_i})(\overline{g}) \text{ for all } g \in \Gamma_{t_i,U^\eta}.$$

Given the form of the constant term this immediately implies that it lies in the -1 eigenspace for the involution induced by complex conjugation.

Recall that in this case $t_i = \begin{pmatrix}\gamma_i b_i & 0\\ 0 & b_i\end{pmatrix}$ for some $\gamma_i \in \hat{\mathcal{O}}^*$ and $b_i \in \mathbf{A}_{F,f}^*$. We will also use that the Ψ_ϕ (which were defined in Section 3.2 as a product of local factors) satisfy

$$\Psi_\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k) = \mu_1(a)\mu_2(d) \text{ for } \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in B(\mathbf{A}), k \in \prod_v \mathrm{SL}_2(\mathcal{O}_v) \subset K_f^s.$$

Note that, in particular, $\Psi_\phi(bg) = \phi_\infty^{-1}(b)\Psi_\phi(g)$ for $b \in B(F) \subset G(\mathbf{A}_f)$.

We are therefore left to show that $\Psi_\phi(\eta_f t_i) = \Psi_{w_0.\phi}(\overline{\eta}_f t_i)$. For this we use the Bruhat decomposition of matrices in $\mathrm{GL}_2(F)$ given by:

$$(5.1) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \begin{pmatrix} 1 & b/d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} & \text{if } c = 0, \\ \begin{pmatrix} 1 & a/c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{ad-bc}{c} & 0 \\ 0 & -c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & d/c \\ 0 & 1 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Since $\Psi_\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \Psi_\phi(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix})\Psi_\phi(g)$ we can consider separately the cases

(a) $\eta = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ for $a, b, d \in F$ and

(b) $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ for $e \in F$.

We check that for (a)

$$\Psi_\phi(\eta_f \begin{pmatrix} \gamma_i b_i & 0 \\ 0 & b_i \end{pmatrix}) = \mu_1(\gamma_i b_i)\mu_2(b_i)\Psi_\phi(\eta_f)$$

and

$$\Psi_{w_0.\phi}(\overline{\eta}_f \begin{pmatrix} \gamma_i b_i & 0 \\ 0 & b_i \end{pmatrix}) = \mu_2(\gamma_i)|\gamma_i|\mu_1(b_i)\mu_2(b_i)\Psi_{w_0.\phi}(\overline{\eta}_f).$$

Since $\gamma_i \in \hat{\mathcal{O}}^*$ and $\chi = \mu_1/\mu_2$ is unramified it suffices to show that $\Psi_\phi(\eta_f) = \Psi_{w_0.\phi}(\overline{\eta}_f)$. In case (b) we similarly reduce to this assertion.

In (a) we get $\Psi_\phi(\eta_f) = \mu_{1,\infty}^{-1}(a)\mu_{2,\infty}^{-1}(d) = \frac{d}{a}$. Since $w_0.\phi$ has infinity type $(\overline{z}, \overline{z}^{-1})$ this equals $\Psi_{w_0.\phi}(\overline{\eta}_f)$. In (b) we need to calculate the Iwasawa decomposition of η in

$\mathrm{GL}_2(F_v)$ if $e \notin \mathcal{O}_v$ (at all other places $\Psi_\phi(\eta_v) = \Psi_{w_0.\phi}(\overline{\eta}_{\overline{v}}) = 1$). It is given by

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix} \begin{pmatrix} -1 & 0 \\ -e^{-1} & -1 \end{pmatrix}.$$

So, if $e \notin \mathcal{O}_v$ then $\Psi_\phi(\eta_v) = (\mu_2/\mu_1)_v(e) = \chi_v^{-1}(e)$, which we claim matches $\Psi_{w_0.\phi}(\overline{\eta}_{\overline{v}}) = (\mu_1/\mu_2)_{\overline{v}}(\overline{e})|\overline{e}|_{\overline{v}}^{-2}$. This follows from $\chi^c = \overline{\chi}$ and $\chi\overline{\chi} = |\cdot|^2$ (for the latter see Lemma 3.14).

Case (2). We now treat the case of $\Gamma_{t_i} = H(\mathfrak{b})$, where the involution is induced by $H \mapsto A\overline{H}A^{-1}$ for $A = \begin{pmatrix} 0 & 1 \\ -N^{-1} & 0 \end{pmatrix}$ with $N = \mathrm{Nm}(\mathfrak{b})$. Considering the effect of the involution on the cusp corresponding to B^η, we claim that

$$\mathcal{G}(\omega_0(\Psi_\phi)^{t_i}_{B^\eta})(g) = \mathcal{G}(\omega_0(\Psi_{w_0.\phi})^{t_i}_{B^{\overline{\eta}A^{-1}}})(A\overline{g}A^{-1}) \text{ for all } g \in \Gamma_{t_i,U^\eta}.$$

Recall that $t_i = \begin{pmatrix} x_ib_i & 0 \\ 0 & b_i \end{pmatrix}$ for some $x_i, b_i \in \mathbf{A}^*_{F,f}$. We have to show that

$$\Psi_\phi(\eta_f t_i) = \Psi_{w_0.\phi}(\overline{\eta}_f A^{-1} t_i). \tag{5.2}$$

Again making use of the Bruhat decomposition, we need to only consider η as in cases (a) and (b) above. Following the arguments used for Case (1), Case(a) reduces immediately to showing that $\Psi_\phi(t_i) = \Psi_{w_0.\phi}(A^{-1}t_i)$. The left hand side equals $\mu_{1,f}(x_ib_i)\mu_{2,f}(b_i)$, the right hand side is

$$\begin{aligned}\Psi_{w_0.\phi}(\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_ib_i & 0 \\ 0 & b_i \end{pmatrix}) &= N^{-1}\Psi_{w_0.\phi}(\begin{pmatrix} b_i & 0 \\ 0 & x_ib_i \end{pmatrix}) \\ &= N^{-1}\mu_{1,f}(x_ib_i)\mu_{2,f}(b_i)|x_i|_f^{-1}.\end{aligned}$$

Equality follows from $|x_i|_f^{-1} = \mathrm{Nm}(\mathfrak{b})$.

For (b), one quickly checks that for $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ the two sides in (5.2) agree.

For the general $\eta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ one shows that, on the one hand,

$$\eta_f \begin{pmatrix} x_ib_i & 0 \\ 0 & b_i \end{pmatrix} = \begin{pmatrix} b_i & 0 \\ 0 & x_ib_i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & ex_i \\ 0 & 1 \end{pmatrix},$$

and on the other hand,

$$\overline{\eta}_f A^{-1}\begin{pmatrix} x_ib_i & 0 \\ 0 & b_i \end{pmatrix} = \begin{pmatrix} x_ib_i & 0 \\ 0 & b_iN \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & \overline{e}x_i/N \\ 0 & 1 \end{pmatrix}.$$

Since $(x_i\overline{x}_i) = (N)$ the valuations of $\overline{e}x_i/N$ agrees with that of $\overline{ex}_i$. Repeating the calculation for $\eta = w_0$ and then applying the argument from Case 1(b) (since χ is unramified we are only concerned about the valuation of the upper right hand entry) we also obtain equality.

□

CHAPTER VI

Bounding the Eisenstein ideal

After defining an Eisenstein ideal in a Hecke algebra acting on cohomological cuspidal automorphic forms, we put the results of Chapters III-V together to prove a bound for its index in terms of a special L-value.

Recall our setup: Let F be an imaginary quadratic field, distinct from $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$. Suppose $\mathfrak{p}$ is a prime of F such that the underlying rational prime p is greater than 3 and splits in F.

We consider unramified Hecke characters $\chi : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of infinity type z^2 (i.e. $\chi_\infty(z) = z^2$). We gave examples of such characters in Section 3.5 and showed that they are anticyclotomic, meaning that they satisfy $\chi^c = \overline{\chi}$. In Corollary 4.18 we obtained characters $\mu_1, \mu_2 : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of infinity type z and z^{-1}, respectively, such that $\chi = \mu_1/\mu_2$, for which we could bound from below the denominator of the Eisenstein cohomology class

$$[\mathrm{Eis}(\omega_0(\phi, \Psi_\phi))] \in H^1(S_{K_f^s}, F_\chi),$$

where $\phi = (\mu_1, \mu_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$. For the definition of $\mathrm{Eis}(\omega_0(\phi, \Psi_\phi))$ and K_f^s see Sections 3.2 and 3.3. We recall that F_χ is the finite extension of $F_\mathfrak{p}$ obtained by adjoining the values of the finite part of both μ_i and $L^{\mathrm{alg}}(0, \chi)$ (cf. Theorem 2.1) and that we call its ring of integers $\mathcal{O}_\chi$. The choice of characters in the proof of Corollary 4.18 is $\mu_1 = \chi \cdot \mu_G$ and $\mu_2 = \mu_G$, where μ_G is Greenberg's character from Lemma 3.18. The character μ_G has conductor (from now on denoted by $\mathfrak{M}_1$) divisible precisely by the primes ramified in F, and at these primes its restriction to the local units has order 2. This also ensures that 1 is the only unit in $\mathcal{O}^*$ congruent to 1 modulo the conductors of μ_i, so that $K_f^s \cap G(\mathbf{Q}) = \mathrm{SL}_2(\mathcal{O})$ (cf. Assumption 5.24). We denote by S the set of places where μ_G is ramified.

6.1 Diamond operators

We assume now in addition that p does not divide $\#\mathrm{Cl}(F)$. Let H be the p-Sylow subgroup of the ray class group $\mathrm{Cl}_{\mathfrak{M}_1}(F) \cong F^* \backslash \mathbf{A}_F^* / \mathbf{C}^* U(\mathfrak{M}_1)$. Since p does not divide the class number of F we have

$$(6.1) \qquad H \cong \prod_{v \in S} H_v \subset \prod_{v \in S} \mathcal{O}_v^*/(1+\mathfrak{M}_{1,v}) \cong (\mathcal{O}/\mathfrak{M}_1)^*$$

for H_v the p-Sylow subgroups of $\mathcal{O}_v^*/(1+\mathfrak{M}_{1,v})$. We define a compact open subgroup $K_f^{H,s}$ containing K_f^s by

$$K_f^{H,s} := \prod_{v \notin S} \mathrm{GL}_2(\mathcal{O}_v) \prod_{v \in S} U^{H_v}(\mathfrak{M}_{1,v}),$$

where

$$U^{H_v}(\mathfrak{M}_{1,v}) = \{k \in \mathrm{GL}_2(\mathcal{O}_v) : \det(k) \in H_v \mod \mathfrak{M}_{1,v}\}.$$

Since the spherical vector $\Psi_{\phi_v}^0$ defined in (3.3) is right-invariant under $U^{H_v}(\mathfrak{M}_{1,v})$ due to $\mu_{1,v}|_{\mathcal{O}_v^*}$ having order 2 we see that $\omega_0(\phi, \Psi_\phi)$ also defines a nontrivial class in $\tilde{H}^1(\partial \tilde{S}_{K_f^{H,s}}, \mathcal{O}_\chi)$ for the slightly larger group $K_f^{H,s}$ (cf. the corresponding statement for K_f^s in Proposition 3.23) and $\mathrm{Eis}(\omega_0(\phi, \Psi_\phi))$ defines a class in $H^1(S_{K_f^{H,s}}, F_\chi)$ which we will denote by c_χ. Our arguments in chapters III-V apply to $K_f^{H,s}$ and this particular character $\phi = (\chi \cdot \mu_G, \mu_G)$ without change. Note, in particular, that $K_f^{H,s} \cap G(\mathbf{Q}) = \mathrm{SL}_2(\mathcal{O})$ also, since $-1 \notin H$ (cf. Section 5.4).

By [U98] §1.2 and §1.4.5 we have for any $\mathcal{O}$-algebra R an R-linear action of the ray class group $\mathrm{Cl}_{\mathfrak{M}_1}(F)$ on $H^1(S_{K_f^{H,s}}, R)$ via the diamond operators (see Section 2.9.2). For a prime ideal $\mathfrak{q}$ not dividing $\mathfrak{M}_1$ the action of $[\mathfrak{q}] \in \mathrm{Cl}_{\mathfrak{M}_1}(F)$ is given by the Hecke operator

$$S_v = [K_f^{H,s} \begin{pmatrix} \pi_v & 0 \\ 0 & \pi_v \end{pmatrix} K_f^{H,s}]$$

for v the place corresponding to $\mathfrak{q}$ and π_v a uniformizer of $\mathcal{O}_v$. This uses that

$$K_f^{H,s} \supset K(\mathfrak{M}_1) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K_f^0 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod \mathfrak{M}_1 \right\}.$$

We claim that the action of the diamond operators S_v, $v \notin S$, is trivial if $[(\pi_v)] \in H \subset \mathrm{Cl}_{\mathfrak{M}_1}(F)$ (recall that we use the notation (g) for the fractional ideal generated by a finite idele g). By (6.1) we have $[(\pi_v)] = [\alpha\mathcal{O}]$ for some $\alpha \in \mathcal{O}$ with $\alpha \in H \mod \mathfrak{M}_1$. Therefore S_v has the same action as $[K_f^{H,s} \begin{pmatrix} \alpha' & 0 \\ 0 & \alpha' \end{pmatrix} K_f^{H,s}]$ for $\alpha' \in \mathbf{A}_F^*$ an idele with $(\alpha'_f) = \alpha\mathcal{O}$ and $\alpha'_v = 1$ for each $v \in S$ and $v = \infty$. But $\begin{pmatrix} \alpha' & 0 \\ 0 & \alpha' \end{pmatrix} \in G(\mathbf{A})$ can be written as the product of $\begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \in Z(\mathbf{Q}) \hookrightarrow G(\mathbf{A})$ and an element of $Z_\infty \cdot K_f^{H,s}$ and so its Hecke action is trivial.

We conclude that we have an action of $\mathcal{O}_\chi[\mathrm{Cl}_{\mathfrak{M}_1}(F)/H]$ on $H^1(S_{K_f^{H,s}}, R)$ for any $\mathcal{O}_\chi$-algebra R. Since $\mathrm{Cl}_{\mathfrak{M}_1}(F)/H$ has order prime to p, $\mathcal{O}_\chi[\mathrm{Cl}_{\mathfrak{M}_1}(F)/H]$ is semisimple. For $\omega := \mu_1\mu_2$, which can be viewed as a character of $\mathrm{Cl}_{\mathfrak{M}_1}(F)/H$, let e_ω be the idempotent associated to ω^{-1}, so that $S_v e_\omega = \omega(\pi_v)^{-1} e_\omega$. For $R = \mathbf{C}$ the idempotent e_ω projects to cuspforms with central character ω via the Eichler-Shimura-Harder isomorphism.

6.2 Main result

Recall that $H_!^i(S_{K_f^{H,s}}, R) := \mathrm{im}(H_c^1(S_{K_f^{H,s}}, R) \to H^1(S_{K_f^{H,s}}, R))$ for any $\mathcal{O}_\chi$-algebra R and that $\tilde{H}_!^1(S_{K_f^{H,s}}, \mathcal{O}_\chi) := \mathrm{im}(H_!^1(S_{K_f^{H,s}}, \mathcal{O}_\chi) \to H_!^1(S_{K_f^{H,s}}, F_\chi))$.

Definition 6.1. Denote by $\mathbf{T}_\chi$ the $\mathcal{O}_\chi$-subalgebra of $\mathrm{End}_{\mathcal{O}_\chi}(e_\omega \tilde{H}_!^1(S_{K_f^{H,s}}, \mathcal{O}_\chi))$ generated by the Hecke operators $T_v = [K_f^{H,s} \begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix} K_f^{H,s}]$ for all primes $v \notin S$.

Definition 6.2 (Eisenstein ideal). We call the ideal $\mathbf{I}_{\mu_1,\mu_2} \subseteq \mathbf{T}_\chi$ generated by

$$\left\{T_v - \mu_{1,v}^{-1}(\mathfrak{P}_v) - \mu_{2,v}^{-1}(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v) | v \notin S\right\}$$

the *Eisenstein ideal associated to* (μ_1, μ_2). Here $\mathfrak{P}_v$ is the maximal ideal in $\mathcal{O}_v$.

Our main result can now be stated as:

Theorem 6.3. *There is an* $\mathcal{O}_\chi$*-algebra surjection*

$$\mathbf{T}_\chi / \mathbf{I}_{\mu_1,\mu_2} \twoheadrightarrow \mathcal{O}_\chi / \left(L^{\mathrm{alg}}(0,\chi)\right).$$

Proof. Recall the long exact sequence

$$\ldots \to H^1_c(S_{K_f^{H,s}}, R) \to H^1(S_{K_f^{H,s}}, R) \xrightarrow{\mathrm{res}} H^1(\partial \overline{S}_{K_f^{H,s}}, R) \to H^2_c(S_{K_f^{H,s}}, R) \to \ldots$$

for any $\mathcal{O}_\chi$-algebra R. The class $c_\chi \in H^1(S_{K_f^{H,s}}, F_\chi)$ is annihilated by $T_v - \mu_{1,v}^{-1}(\mathfrak{P}_v) - \mu_{2,v}^{-1}(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v)$, $v \notin S$ (cf. Lemma 3.11). Since $\chi^c = \overline{\chi}$, its (non-trivial) restriction to the boundary $\mathrm{res}(c_\chi)$ lies in $\tilde{H}^1(\partial \overline{S}_{K_f^{H,s}}, \mathcal{O}_\chi)$ (cf. Lemma 3.24). In Corollary 4.18 we showed that the denominator of c_χ is bounded below by $L^{\mathrm{alg}}(0,\chi)$ (i.e., $\delta(c_\chi) \subset (L^{\mathrm{alg}}(0,\chi))$. In Chapter V, we proved that there exists $c \in \tilde{H}^1(S_{K_f^{H,s}}, \mathcal{O}_\chi)$ with the same restriction to the boundary as the Eisenstein cohomology class c_χ. Note that

$$\mathrm{res}(e_\omega c) = e_\omega \mathrm{res}(c) = e_\omega \mathrm{res}(c_\chi) = \mathrm{res}(c_\chi)$$

since

$$S_v(c_\chi) = \omega^{-1}(\pi_v) c_\chi.$$

We can now prove the theorem following the proof of Proposition 6.2 in [S02a]: Without loss of generality, we can assume that $\delta(c_\chi) \subsetneq \mathcal{O}_\chi$; there is nothing to prove otherwise. Let δ be a generator of $\delta(c_\chi)$. Then δc_χ is an element of an $\mathcal{O}_\chi$-basis of $\mathrm{e}_\omega \tilde{H}^1(S_{K_f^{H,s}}, \mathcal{O}_\chi)$. By construction, $c_0 := \delta \cdot (e_\omega c - c_\chi) \in \tilde{H}^1_!(S_{K_f^{H,s}}, \mathcal{O}_\chi)$ is a nontrivial element of an $\mathcal{O}_\chi$-basis of $\mathrm{e}_\omega \tilde{H}^1_!(S_{K_f^{H,s}}, \mathcal{O}_\chi)$. Extend c_0 to an $\mathcal{O}_\chi$-basis $c_0, c_1, \ldots c_d$ of $\tilde{H}^1_!(S_{K_f^{H,s}}, \mathcal{O}_\chi)$. For each $t \in \mathbf{T}_\chi$ write

$$t(c_0) = \sum_{i=0}^{d} a_i(t) c_i,\ a_i(t) \in \mathcal{O}_\chi.$$

Then

$$\mathbf{T}_\chi \to \mathcal{O}_\chi/(\delta),\ t \mapsto a_0(t) \mod \delta \tag{6.2}$$

is an $\mathcal{O}_\chi$-module surjection. We claim that it is independent of the $\mathcal{O}_\chi$-basis chosen and that it is a homomorphism of $\mathcal{O}_\chi$-algebras with the Eisenstein ideal $\mathbf{I}_{\mu_1,\mu_2}$ contained in its kernel. To prove this it suffices to check that each $a_0(T_v - \mu_{1,v}^{-1}(\mathfrak{P}_v) - \mathrm{Nm}(\mathfrak{P}_\mathrm{v})\mu_{2,\mathrm{v}}^{-1}(\mathfrak{P}_\mathrm{v}))$, $v \notin S$ is contained in $\delta\mathcal{O}_\chi$. This is an easy calculation. Let $t_v = T_v - \mu_{1,v}^{-1}(\mathfrak{P}_v) - \mathrm{Nm}(\mathfrak{P}_v)\mu_{2,v}^{-1}(\mathfrak{P}_v)$. Then by Lemma 3.11, $t_v c_\chi = 0$ and hence $t_v c_0 = t_v \delta e_\omega c \in \delta \tilde{H}^1_!(S_{K_f^{H,s}}, \mathcal{O}_\chi)$. Thus $a_0(t_v) \in \delta\mathcal{O}_\chi$ and (6.2) is a well-defined $\mathcal{O}_\chi$-algebra surjection, coinciding with $T_v \mapsto \mu_{1,v}^{-1}(\mathfrak{P}_v) + \mathrm{Nm}(\mathfrak{P}_v)\mu_{2,v}^{-1}(\mathfrak{P}_v) \mod \delta$. Since $\mathcal{O}_\chi/(\delta) \twoheadrightarrow \mathcal{O}_\chi/(L^{\mathrm{alg}}(0,\chi))$ by Corollary 4.18, this proves our theorem.

□

Remark 6.4. 1. Note that the right-hand side in the statement of the theorem is nontrivial if p divides $L^{\mathrm{alg}}(0,\chi)$. If the left-hand side $\mathbf{T}_\chi/\mathbf{I}_{\mu_1,\mu_2}$ is non-trivial then a minimal prime contained in $\mathbf{I}_{\mu_1,\mu_2}$ gives rise (via the Eichler-Shimura-Harder isomorphism) to a cuspidal eigenform with eigenvalues congruent to those of the Eisenstein cohomology class. Using the same notation as in the proof, the class $c \in \tilde{H}^1(S_{K_f^{H,s}}, \mathcal{O}_\chi)$ exhibits a non-trivial element of the cohomological congruence module $M/(N \oplus N')$ for $M = \tilde{H}^1(S_{K_f^{H,s}}, \mathcal{O}_\chi)$, $N = \tilde{H}^1_!(S_{K_f^{H,s}}, \mathcal{O}_\chi)$, and $N' = \delta c_\chi \mathcal{O}_\chi$. See [P] and [Gh] for more details on congruence modules and their relationships.

2. $L(0,\chi) \neq 0$ since 0 is the abscissa of convergence for this (non-unitary) infinity type (see [La] XV §4).

3. Observe that for $\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$ (the fields excluded above) the boundary cohomology for constant coefficients and maximal compact open subgroups K_f is trivial.

CHAPTER VII

Application to bounding Selmer groups

In this chapter we will demonstrate how to apply Theorem 6.3 to bound the size of certain Selmer groups from below by $L^{\mathrm{alg}}(0,\chi)$.

7.1 Background

We keep the following notation from the previous Chapter: p, F, $\mathfrak{p}$, χ, F_χ, $\mathcal{O}_\chi$. Let R be the integer ring in a finite extension L of $F_{\mathfrak{p}}$. Denote its maximal ideal by $\mathfrak{m}_R$. Let $G_F := \mathrm{Gal}(\overline{F}/F)$, and for Σ a finite set of places of F containing the places above p let G_Σ be the Galois group of the maximal extension of F unramified at all places not in Σ. For any place v of F, G_v and I_v denote, respectively, the decomposition group and the inertia subgroup for v determined by $\overline{F} \hookrightarrow \overline{F}_v$. Denote by $\chi_{\mathfrak{p}}$ the infinite order $\mathfrak{p}$-adic Galois character associated to χ (see end of Section 2.6) and by $\epsilon : G_F \to \mathbf{Z}_p^*$ the p-adic cyclotomic character defined by the action of G_F on the p-power roots of unity: $g.\xi = \xi^{\epsilon(g)}$ for ξ with $\xi^{p^m} = 1$ for some m.

7.1.1 Fitting ideals

We recall the definition and basic properties of Fitting ideals. For details we refer to the Appendix of [MW]. Let A be a ring and let M be a finitely generated A-module with generators $m_1, \dots, m_n$. Let $f : A^n \twoheadrightarrow M$ be the surjective A-homomorphism defined by $f(e_i) = m_i$ for $i = 1, \dots, n$. Here e_i denotes the ith standard basis vector of A^n. The *Fitting ideal* $\mathrm{Fitt}_A(M)$ of M is the ideal generated by the determinants

$\det(v_1, \ldots, v_n)$ for which the column vectors $v_1, \ldots, v_n$ lie in $\ker(f)$. One checks that this does not depend on the choice of the generators m_i.

The following proposition contains the properties of the Fitting ideal that we will use:

Proposition 7.1. *(i)* $\mathrm{Fitt}_A(M) \subset \mathrm{Ann}_A(M)$.

(ii) For any A-algebra B we have $\mathrm{Fitt}_B(M \otimes_A B) = \mathrm{Fitt}_A(M) \cdot B$.

(iii) For any ideal $\mathfrak{a} \subset A$ *we have* $\mathrm{Fitt}_A(A/\mathfrak{a}) = \mathfrak{a}$.

(iv) If A is a complete local Noetherian ring with maximal ideal $\mathfrak{m}_A$ *and M an A-module of finite length, then*

$$\mathfrak{m}_A^{\mathrm{length}_A(M)} \subset \mathrm{Fitt}_A(M).$$

Remark 7.2. For rings A that are complete discrete valuation rings of residue characteristic p (like $\mathcal{O}_\chi$ or R) we will write $\mathrm{val}_p(\#M)$ instead of $\mathrm{length}_A(M)$.

7.1.2 Galois cohomology

For any profinite group G (e.g. G_F or G_Σ) call N a topological G-module if it is a commutative topological group with a continuous action of G. Given a topological G-module N, define the continuous cohomology groups $H^i(G, N)$ to be the cohomology of the complex defined by continuous cochains (for details see [Ru] Appendix B.2): Let $C^i(G, N) = \{f : G^i \to N \text{ continuous }\}$. For every $i \geq 0$ there is a coboundary map $d_i : C^i(G, N) \to C^{i+1}(G, N)$ and we set $H^1(G, N) := \ker(d_i)/\mathrm{image}(d_{i-1})$. The group $H^0(G, N)$ can be identified with N^G, and for the trivial G-action on N we have $H^1(G, N) = \mathrm{Hom}_{\mathrm{cts}}(G, N)$. Note also that for discrete modules any homomorphism is continuous.

When N has the additional structure of an S-module for some ring S and the G-action is S-linear (call this an $S[G]$-module), the continuous cohomology groups are S-modules.

If $0 \to T' \to T \to T'' \to 0$ is an exact sequence of topological $S[G]$-modules and if there exists a continuous section (a map of sets, not necessarily a homomorphism) from $T'' \to T$, then we call it a topological short exact sequence (cf. [Ru] Appendix B.2, [BW] p. 258).

Proposition 7.3. *If N is a topological $S[G]$-module, then $H^1(G,N)$ classifies (isomorphism classes of) topological short exact sequences (of $S[G]$-modules)*

$$0 \to N \to E \to \mathbf{1} \to 0,$$

i.e., continuous extension classes of $\mathbf{1}$ by N, where $\mathbf{1}$ is the trivial linear representation of G on S (or S/I for some ideal I).

Proof. Given an extension

$$0 \to N \xrightarrow{f} E \xrightarrow{g} \mathbf{1} \to 0$$

let $e \in E$ project onto an S-generator of $\mathbf{1}$. Define a 1-cocycle $c : G \to N$ by $c(\sigma) = f^{-1}(e - \sigma(e))$. Since E is a continuous extension, c is a continuous 1-cocycle. Two isomorphic extensions give rise to the same cohomology class. Note that, in particular, if E is a split extension, i.e., if there exists a G-invariant continuous section h such that $g \circ h = \mathrm{id}_{\mathbf{1}}$, then this construction yields a 1-coboundary. Furthermore, if two extensions give rise to the same cohomology class they are isomorphic (see argument in [Wa] Proposition 4).

Conversely, given a continuous 1-cocycle $c : G \to N$ let E be the S-module $N \oplus S$, where the action of G on N is extended by $\sigma.r = rc(\sigma) \oplus r$ for $r \in S$. Since c is continuous this gives a topological $S[G]$-module. We note that if c is a 1-coboundary, i.e., of the form $\sigma \mapsto \sigma(n) - n$ for some $n \in N$ then $r \in S \mapsto (-rn, r) \in N \oplus S$ splits the extension $E = N \oplus S$. □

7.1.3 Selmer groups

Selmer groups are generalizations of the ideal class groups of number fields. To motivate the general definition of a Selmer group we recall the connection of class groups with Galois groups via class field theory:

Example 7.4. By class field theory we have a canonical identification of the ideal class group $\mathrm{Cl}(F)$ with the Galois group of the maximal everywhere unramified abelian extension of F, the Hilbert class field H_F. The latter is the quotient of the Galois group of the maximal abelian extension F^{ab} over F by the images of all the inertia groups.

If we are just interested in the p-Sylow subgroup $\mathrm{Cl}(F)[p^\infty]$, we can recover this from $\mathrm{Hom}(\mathrm{Gal}(H_F/F), \mathbf{Q}_p/\mathbf{Z}_p)$ by taking the Pontryagin dual $\mathrm{Hom}(\cdot, \mathbf{Q}_p/\mathbf{Z}_p)$. But $\mathrm{Hom}(\mathrm{Gal}(H_F/F), \mathbf{Q}_p/\mathbf{Z}_p)$ can be identified with

$$\ker(H^1(G_F, \mathbf{Q}_p/\mathbf{Z}_p) \to \prod_v H^1(I_v, \mathbf{Q}_p/\mathbf{Z}_p)).$$

This will turn out to be the Selmer group for G_F of the trivial Galois representation $\mathbf{Z}_p$.

Definition 7.5 (Pontryagin dual). For a topological R-module N put $N^\vee = \mathrm{Hom}_{\mathrm{cts}}(N, \mathbf{Q}_p/\mathbf{Z}_p)$. The group $N^\vee$ is an R-module via $r.f(n) = f(rn), r \in R, f \in N^\vee, n \in N$.

Lemma 7.6. *If N is a finite R-module, then*

$$\mathrm{Hom}_R(N, R^\vee) \cong N^\vee,\ f \mapsto (n \mapsto f(n)(1)),$$

is an isomorphism of R-modules.

Proof. (This is Lemma 6.1.1(ii)) from [S04].) Observe that $R^\vee = \cup R^\vee[\mathfrak{m}_R^n]$. Since N is a finite R-module it follows that given $f \in \mathrm{Hom}_R(N, R^\vee)$ there is an n such that $\mathrm{im}(f) \in R^\vee[\mathfrak{m}_R^n]$. Thus the map in the lemma takes values in $N^\vee$ (and not just $\mathrm{Hom}(N, \mathbf{Q}_p/\mathbf{Z}_p)$). It is then easy to check that the map

$$N^\vee \to \mathrm{Hom}_R(N, R^\vee),\ f \mapsto (n \mapsto (r \mapsto f(rn))),$$

and the map in the lemma are inverses of each other. □

By an R-lattice in a finite-dimensional L-space V we mean a finite R-submodule $M \subset V$ that spans V over L.

We define the Selmer group for a "p-ordinary Galois representation" following Greenberg (cf. [G89]): Suppose given an n-dimensional L-space V and a continuous representation $\rho : G_\Sigma \to \mathrm{Aut}_L(V)$, i.e., such that there exists a G_Σ-stable R-lattice $M \subset V$ on which G_Σ acts continuously with respect to the $\mathfrak{m}_R$-adic topology. Suppose also given a G_w-stable subspace V_w^+ for each place w of F lying over p.

Let $M \subset V$ be any G_Σ-stable R-lattice. For each $w|p$ let $M_w^+ = M \cap V_w^+$. This is a G_w-stable R-lattice in V_w^+. The module $M^* = M \otimes_R R^\vee$ is a discrete $R[G_\Sigma]$-module and for $w|p$ the module $M_w^{+,*} := M_w^+ \otimes_R R^\vee$ is a discrete $R[G_w]$-module. Moreover, there are canonical maps $M_w^{+,*} \to M^*$ coming from the inclusions $M_w^+ \hookrightarrow M$. For each finite set $\Sigma' \subset \Sigma \backslash \{w|p\}$ we define a Selmer group associated to M by

Definition 7.7.

$$\mathrm{Sel}(\Sigma', M) = \ker(H^1(G_\Sigma, M^*) \to \oplus_{w|p} H^1(I_w, M^*/M_w^{+,*}) \oplus_{w \in \Sigma'} H^1(I_w, M^*)).$$

We write $\mathrm{Sel}(M)$ for $\mathrm{Sel}(\emptyset, M)$.

Lemma 7.8. *The Pontryagin dual of* $\mathrm{Sel}(T, M)$ *is a finitely generated R-module.*

References. [S04] Corollary 6.1.4, [Ru3] Lemma 1.5.7(iii). □

Definition 7.9. Suppose $\rho : G_\Sigma \to R^*$ is a continuous Galois character for some Σ as above. Denote by $R(\rho)$ the free rank one R-module on which G_Σ acts via ρ.

We will apply the Selmer group definition in the case of 1-dimensional Galois characters arising from Hecke characters of type (A_0):

Example 7.10. For $\lambda : F^* \backslash \mathbf{A}_F \to \mathbf{C}^*$ a Hecke character of type (A_0), i.e., with infinity type $z^a \overline{z}^b$ with $a, b \in \mathbf{Z}$, we associated at the end of Section 2.6 a Galois character

$$\lambda_{\mathfrak{p}} : G_\Sigma^{\mathrm{ab}} \to \mathcal{O}_\lambda^*,$$

where Σ consists of the places dividing the conductor $\mathfrak{f}_\lambda$ and the places dividing p, and $\mathcal{O}_\lambda$ is the ring of integers in the finite extension F_λ of $F_{\mathfrak{p}}$ containing the values of λ_f. Extending $\lambda_{\mathfrak{p}}$ trivially to G_Σ we put $M = \mathcal{O}_\lambda(\lambda_{\mathfrak{p}})$ and $V = F_\lambda(\lambda_{\mathfrak{p}}) := M \otimes_{\mathcal{O}_\lambda} F_\lambda$. By definition $\lambda_{\mathfrak{p}}$ is "locally algebraic", i.e., for each $w|p$ and $u \in \mathcal{O}_w^*$ with $u \equiv 1$ mod $\mathfrak{f}_\lambda \mathcal{O}_w$ we have $\lambda_{\mathfrak{p}}(\mathrm{rec}(u)) = u^a \overline{u}^b$, where rec is the Artin reciprocity map. By a theorem of Tate (cf. [Se68] III A7) this implies that the local Galois representations $\lambda_{\mathfrak{p}}|_{G_w}$ are Hodge-Tate, i.e., $\widehat{\overline{F}}_\lambda(\lambda_{\mathfrak{p}}|_{G_{\mathfrak{p}}}) \cong \widehat{\overline{F}}_\lambda(\epsilon^{-a})$ and $\widehat{\overline{F}}_\lambda(\lambda_{\mathfrak{p}}|_{G_{\overline{\mathfrak{p}}}}) \cong \widehat{\overline{F}}_\lambda(\epsilon^{-b})$. The exponents $-a$ and $-b$ are called the "Hodge-Tate weights" of the representation at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, respectively.

Remark 7.11. We use here the arithmetic Frobenius normalization in the Artin reciprocity map which implies that $\epsilon(\mathrm{rec}(u)) = u^{-1}$ for $u \in \mathcal{O}_{\mathfrak{p}}^*$. This choice was implicitly taken in our definition of the L-function $L(s,\lambda)$ in Section 2.6, which equals the Artin L-function $L(s,\lambda_{\mathfrak{p}})$ under this normalization.

For Hecke characters λ of infinity type $z^a\overline{z}^b$ with $a,b \in \mathbf{Z}$ and $V = F_\lambda(\lambda_{\mathfrak{p}})$ we let

$$V_{\mathfrak{p}}^+ = \begin{cases} V & \text{if } a < 0 \text{ (or HT-wt} > 0), \\ \{0\} & \text{if } a \geq 0 \text{ (or HT-wt} \leq 0) \end{cases}$$

and

$$V_{\overline{\mathfrak{p}}}^+ = \begin{cases} V & \text{if } b < 0, \\ \{0\} & \text{if } b \geq 0. \end{cases}$$

The Hodge-Tate weights of the Galois characters we will be interested in are summarized in the following table:

HT-wt at	$\mathfrak{p}$	$\overline{\mathfrak{p}}$
ϵ	1	1
$\mu_{1,\mathfrak{p}}$	-1	0
$\mu_{2,\mathfrak{p}}$	1	0
$\chi_{\mathfrak{p}}$	-2	0
$\chi_{\mathfrak{p}}\epsilon$	-1	1
$\chi_{\mathfrak{p}}^{-1}\epsilon^{-1}$	1	-1

Example 7.12 (Continuation of Example 7.4). In our general setup we can recover the class group example by taking $M = \mathbf{Z}_p$ and $\Sigma = \{w|p\}$. As explained in the earlier example we get $\mathrm{Sel}(\mathbf{Z}_p)^\vee \cong \mathrm{Cl}(F)[p^\infty]$.

For a finite extension K over F the Galois group $\mathrm{Gal}(K/F)$ acts on $\mathrm{Cl}(K)$. If we want to study this finer structure, we can do the following: Let $\chi : \Delta = \mathrm{Gal}(K/F) \to R^*$ be a finite order character, with K an abelian extension of F of degree prime to p. For a $\mathbf{Z}_p[\Delta]$-module B (e.g. $\mathrm{Cl}(K)[p^\infty]$) denote by B^χ the χ-isotypical piece, i.e. $B^\chi := \{b \in B \otimes_{\mathbf{Z}_p} R : \gamma.b = \chi(\gamma)b \text{ for every } \gamma \in \Delta\}$. Take Σ sufficiently large such that Δ is a quotient of G_Σ and extend χ trivially to G_Σ. We claim that

$$\mathrm{Sel}(\Sigma\backslash\{w|p\}, R(\chi))^\vee \cong \mathrm{Cl}(K)[p^\infty]^\chi.$$

This can be seen as follows: In our general notation we have $M = R(\chi)$ and $\Sigma' = \Sigma\backslash\{w|p\}$. Since χ has finite order, its Hodge-Tate weight is 0 and $M_w^+ = 0$ for all $w|p$. Therefore, by definition

$$\mathrm{Sel}(\Sigma', R(\chi)) = \ker(H^1(G_F, M^*) \to \prod_v H^1(I_v, M^*)).$$

Since $[K : F]$ is prime to p the inflation-restriction sequence (see [Ru] Prop. B.2.5) implies that $H^1(G_F, M^*) \cong H^1(G_K, M^*)^\Delta$. Together this shows that $\mathrm{Sel}(\Sigma', R(\chi)) = \mathrm{Hom}(\mathrm{Gal}(H_K/K), M^*)^\Delta = \mathrm{Hom}(\mathrm{Cl}(K)[p^\infty]^\chi, R^\vee)$ for the Hilbert class field H_K of K. The claim follows together with Lemma 7.6.

Definition 7.13. A Galois character $\rho : G_F \to R^*$ is called anticyclotomic if it satisfies $\rho^c = \rho^{-1}$, where $\rho^c(\sigma) = \rho(c\sigma c^{-1})$ with $c \in G_{\mathbf{Q}}$ a lift of the non-trivial automorphism of $\mathrm{Gal}(F/\mathbf{Q})$.

Example 7.14. If $\lambda : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ is an anticyclotomic unitary Hecke character (i.e. $\lambda^c = \overline{\lambda} = \lambda^{-1}$), then $\lambda_{\mathfrak{p}}$ is anticyclotomic.

Lemma 7.15. *We have* $\mathrm{Sel}(\rho) \cong \mathrm{Sel}(\rho^c)$. *Here the isomorphism is induced by conjugation of* G_F *by a lift of the non-trivial automorphism of Gal*$(F/\mathbf{Q})$. □

7.2 Statement and discussion of result

Let Σ_0 be the set comprising the places above p and the places ramified in $F/\mathbf{Q}$. Using a method developed by Wiles, Skinner, and Urban, we will prove:

Proposition 7.16. *With the same assumptions and notation as Theorem 6.3 and $\Sigma = \Sigma_0$, we have*

$$\mathrm{val}_p(\#\mathrm{Sel}(M)^\vee) \geq \mathrm{val}_p(\#(\mathbf{T}_\chi/\mathbf{I}_{\mu_1,\mu_2}))$$

for either $M = \mathcal{O}_\chi(\chi_\mathfrak{p}\epsilon)$ or $M = \mathcal{O}_\chi((\chi_\mathfrak{p}\epsilon)^c) = \mathcal{O}_\chi(\chi_\mathfrak{p}^{-1}\epsilon^{-1})$ (the two Selmer groups are isomorphic by Lemma 7.15).

Remark 7.17. Note that $\chi_\mathfrak{p}\epsilon$ has negative Hodge-Tate weight at the place $\mathfrak{p}$ and positive at the place $\overline{\mathfrak{p}}$, so $\mathrm{Sel}(\mathcal{O}_\chi(\chi_\mathfrak{p}\epsilon))$ will consist of cohomology classes unramified at $\mathfrak{p}$, but possibly ramified at $\overline{\mathfrak{p}}$.

Using Theorem 6.3, this immediately gives us lower bounds on the size of the Selmer groups in terms of the special L-value:

Corollary 7.18. *With the same assumptions and notation as Theorem 6.3 and $\Sigma = \Sigma_0$, we have*

$$\mathrm{val}_p(\#\mathrm{Sel}(M)^\vee) \geq \mathrm{val}_p(\#(\mathcal{O}_\chi/(L^{\mathrm{alg}}(0,\chi))))$$

for $M = \mathcal{O}_\chi(\chi_\mathfrak{p}\epsilon)$ or $M = \mathcal{O}_\chi((\chi_\mathfrak{p}\epsilon)^c) = \mathcal{O}_\chi(\chi_\mathfrak{p}^{-1}\epsilon^{-1})$.

Remark 7.19. That $L^{\mathrm{alg}}(0,\chi)$ gives bounds for these two Selmer groups is related (via the anticyclotomic main conjecture) to the fact that $L^{\mathrm{alg}}(0,\chi)$ gets interpolated by two p-adic L-functions, see [AH] p. 12. See Remark 7.35 for the relation of our results to consequences of the Main Conjecture of Iwasawa theory.

7.3 Proof of Proposition 7.16

We need to procure the ingredients of the following proposition, adapted for our purposes from [S04] Proposition 6.1.17:

Proposition 7.20. *Let $\rho : G_\Sigma \to \mathcal{O}_\chi^*$ be a continuous 1-dimensional representation with positive Hodge-Tate weight at one of the primes lying above p, negative at the other (call the latter prime w). Denote the module by M and write $\overline{\rho}$ for the reduction modulo the maximal ideal of $\mathcal{O}_\chi$. Let T be a finite $\mathcal{O}_\chi$-algebra and $I \subset T$ an ideal such that the $\mathcal{O}_\chi$-algebra structure map surjects onto T/I.*

Suppose we are given:

- *a finite* T*-module* $\mathcal{L}$ *on which* G_Σ *acts continuously and* T*-linearly and having no* $T[G_\Sigma]$ *-quotient isomorphic to* $\overline{\rho}$*.*
- *a* $T[G_\Sigma]$*-submodule* $\mathcal{L}_1 \subset \mathcal{L}$ *such that* G_Σ *acts trivially on* $\mathcal{L}/\mathcal{L}_1$ *and* $\mathcal{L}/\mathcal{L}_1 \cong T/I$*.*
- *a finite* T*-module* $\mathcal{T}$ *with* $\mathrm{Fitt}_T(\mathcal{T}) \subset I \subset \mathrm{Ann}_T(\mathcal{T})$ *and a* $T[G_\Sigma]$*-identification* $\mathcal{L}_1 \cong M \otimes_{\mathcal{O}_\chi} \mathcal{T}$*.*

We further require that the $T[G_\Sigma]$*-extension*

$$0 \to \mathcal{L}_1 \to \mathcal{L} \to \mathcal{L}/\mathcal{L}_1 \to 0$$

be split when viewed as a $T[I_w]$*-extension. Given this set-up,*

$$\mathrm{val}_p(\#\mathrm{Sel}(M)^\vee) \geq \mathrm{val}_p(\#(T/I)).$$

Proof. We redo the proof of [S04] in our special case. As in Proposition 7.3 we fix $e \in \mathcal{L}$ projecting onto a T-generator of $\mathcal{L}/\mathcal{L}_1$ and define a 1-cocycle $c : G_\Sigma \to M \otimes \mathcal{T}$ by

$$c(\sigma) = \text{ the image of } e - \sigma(e) \text{ in } \mathcal{L}_1.$$

Consider the $\mathcal{O}_\chi$-homomorphism

$$\phi : \mathrm{Hom}_{\mathcal{O}_\chi}(\mathcal{T}, \mathcal{O}_\chi^\vee) \to H^1(G_\Sigma, M \otimes \mathcal{O}_\chi^\vee),\ \phi(f) = \text{ the class of } (1 \otimes f) \circ c.$$

We will show that

(i) $\mathrm{im}(\phi) \subset \mathrm{Sel}(M)$,

(ii) $\ker(\phi)^\vee = 0$.

From (i) it follows that

$$\mathrm{val}_p(\#\mathrm{Sel}(M)^\vee) \geq \mathrm{val}_p(\#\mathrm{im}(\phi)^\vee).$$

From (ii) it follows that

$$\begin{aligned}\mathrm{val}_p(\#\mathrm{im}(\phi)^\vee) &\geq \mathrm{val}_p(\#\mathrm{Hom}(\mathrm{Hom}_{\mathcal{O}_\chi}(\mathcal{T}, \mathcal{O}_\chi^\vee), \mathbf{Q}_p/\mathbf{Z}_p)) \\ &= \mathrm{val}_p(\#\mathrm{Hom}(\mathcal{T}^\vee, \mathbf{Q}_p/\mathbf{Z}_p)) \\ &= \mathrm{val}_p(\#\mathcal{T}) \\ &\geq \mathrm{val}_p(\#T/I),\end{aligned}$$

where the first equality comes from Lemma 7.6, and the last inequality from our assumption on $\mathcal{T}$ and Proposition 7.1(iv). The latter implies that $\mathrm{length}_T(\mathcal{T}) \geq \mathrm{length}_T(T/I)$. Since the action of T on both $\mathcal{T}$ and T/I is via T/I, which is a quotient of $\mathcal{O}_\chi$, any $\mathcal{O}_\chi$-submodule of $\mathcal{T}$ or T/I is, in fact, a T-submodule. This implies the corresponding inequality for the $\mathcal{O}_\chi$-lengths.

For (i) we observe that the assumption that the extension splits when considered as an extension of $T[I_w]$-modules implies by Proposition 7.3 that the class c_w in $H^1(I_w, \mathcal{L}_1)$ determined by c is the zero class. This shows that $\mathrm{im}(\phi) \subset \mathrm{Sel}(M)$.

To prove (ii) we first observe that for any $f \in \mathrm{Hom}_{\mathcal{O}_\chi}(\mathcal{T}, \mathcal{O}_\chi^\vee)$, $\ker(f)$ has finite index in $\mathcal{T}$ (like in the proof of Lemma 7.6 we see that $f \in \mathrm{Hom}_{\mathcal{O}_\chi}(\mathcal{T}, \mathcal{O}_\chi^\vee[\mathfrak{p}^n])$ for some n). Suppose now that $f \in \ker(\phi)$. We claim that the class of c in $H^1(G_\Sigma, M \otimes_{\mathcal{O}_\chi} \mathcal{T}/\ker(f))$ is zero. To see this, let $X = \mathcal{O}_\chi^\vee/\mathrm{im}(f)$ and observe that there is an exact sequence

$$H^0(G_\Sigma, M \otimes_{\mathcal{O}_\chi} X) \to H^1(G_\Sigma, M \otimes_{\mathcal{O}_\chi} \mathcal{T}/\ker(f)) \to H^1(G_\Sigma, M \otimes_{\mathcal{O}_\chi} \mathcal{O}_\chi^\vee).$$

Since $f \in \ker(\phi)$ and the second arrow in the sequence comes from the inclusion $\mathcal{T}/\ker(\phi) \hookrightarrow \mathcal{O}_\chi^\vee$ induced by f, the image in the right module of the class of c in the middle is zero. Our claim follows therefore if the module on the left is trivial. But the dual of this module is a subquotient of $\mathrm{Hom}_{\mathcal{O}_\chi}(M, \mathcal{O}_\chi)$ on which G_Σ acts trivially. By assumption, however, M has no nontrivial subquotients.

Suppose in addition that f is non-trivial, i.e. $\ker(f) \subsetneq \mathcal{T}$. Then there exists a T-module A with $\ker(f) \subset A \subset \mathcal{T}$ such that $\mathcal{T}/A \cong \mathcal{O}_\chi/\mathfrak{p}$ (we use here again that any $\mathcal{O}_\chi$-submodule of $\mathcal{T}$ is actually a T-submodule). From our claim it follows now that the $T[G_\Sigma]$-extension

$$0 \to M \otimes_{\mathcal{O}_\chi} \mathcal{O}_\chi/\mathfrak{p} \cong M \otimes_{\mathcal{O}_\chi} \mathcal{T}/A \to \mathcal{L}/(M \otimes_{\mathcal{O}_\chi} A) \to \mathcal{L}/\mathcal{L}_1 \to 0$$

is split. But this would give a $T[G_\Sigma]$-quotient of $\mathcal{L}$ isomorphic to $\overline{\rho}$, which contradicts our assumption. Hence $\ker(\phi)$ (and therefore also $\ker(\phi)^\vee$) are trivial. □

We will apply this Proposition for $\rho = \chi_\mathfrak{p}\epsilon$ (respectively $\rho = (\chi_\mathfrak{p}\epsilon)^c$), T a localization of $\mathbf{T}_\chi$ and I the ideal corresponding to $\mathbf{I}_{\mu_1,\mu_2}$. In the following we demonstrate how to obtain $\mathcal{L}$ and $\mathcal{L}_1$ from the Galois representations associated to cuspidal automorphic representations by the work of Taylor *et al.*

7.3.1 Galois representations attached to cuspidal automorphic representations

Recall the Hecke-equivariant Eichler-Shimura-Harder isomorphism

$$e_\omega H^1_!(\overline{S}_{K_f}, \mathbf{C}) \cong S_0(K_f, \omega, \mathbf{C})$$

(Theorem 2.6 and the end of Section 6.1). Here $S_0(K_f, \omega, \mathbf{C})$ denotes the space of cuspidal automorphic forms of $\mathrm{GL}_2(F)$ of weight 0, right-invariant under $K_f \subset G(\mathbf{A}_f)$ with central character ω (see Section 2.7). This was isomorphic to $\oplus\pi_f^{K_f}$ for automorphic representations π of a certain infinity type with central character ω.

Combining the work of Taylor, Harris and Soudry with results of Friedberg-Hoffstein and Laumon/Weissauer, one can show the following:

Theorem 7.21. *Given a cuspidal automorphic representation π with π_∞ isomorphic to the principal series representation corresponding to*

$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto \left(\frac{t_1}{|t_1|}\right)\left(\frac{|t_2|}{t_2}\right)$$

and cyclotomic central character ω (i.e. $\omega^c = \omega$), let Σ_π denote the set of places above p, the primes where π or π^c is ramified, and primes ramified in $F/\mathbf{Q}$.

Then there exists a continuous Galois representation $\rho_\pi : Gal(\overline{F}/F) \to \mathrm{GL}_2(\overline{F}_\mathfrak{p})$ such that if $v \notin \Sigma_\pi$, then ρ_π is unramified at v and the characteristic polynomial of

$\rho_f(\mathrm{Frob}_v)$ is $x^2 - a_v(\pi)x + \omega^{-1}(\mathfrak{P}_v)\mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{P}_v)$, where $a_v(\pi)$ is the Hecke eigenvalue corresponding to T_v. The image of the Galois representation is actually inside $\mathrm{GL}_2(L)$ for a finite extension L of $F_{\mathfrak{p}}$ and the representation is absolutely irreducible.

Remark 7.22. 1. Taylor relates π to Siegel modular forms via theta lifts and uses the Galois representations associated to Siegel modular forms to find ρ_π.

2. Taylor had some additional technical assumption in [T2] and only showed the equality of Hecke and Frobenius polynomial outside a set of places of zero density. For this strengthening of Taylor's result see [BHR].

3. Since Taylor's convention for the Hecke operators differs from ours, the Galois representations as stated above are twists of Taylor's Galois representation by the central character.

Urban studied in [U98] the case of ordinary automorphic representations π, and together with results in [U04] on the Galois representations attached to ordinary Siegel modular forms shows that for these π one has a particularly nice form for ρ_π when restricted to the decomposition group of $\mathfrak{p}$:

Theorem 7.23 (Corollaire 2 of [U04]). *If π is a cuspidal automorphic representation with cyclotomic central character and is ordinary at $\mathfrak{p}$ (i.e., π unramified at $\mathfrak{p}$ and $|a_{\mathfrak{p}}(\pi)|_p = 1$), then the Galois representation ρ_π is ordinary and*

$$\rho_\pi|_{D_{\mathfrak{p}}} \cong \begin{pmatrix} \Psi_1 & * \\ 0 & \Psi_2 \end{pmatrix},$$

where $\Psi_2|_{I_{\mathfrak{p}}} = 1$, and $\Psi_1|_{I_{\mathfrak{p}}} = \det(\rho_\pi)|_{I_{\mathfrak{p}}} = \epsilon$.

As indicated at the start of this section the connection to our work in the previous chapters will come via the Eichler-Shimura-Harder isomorphism $e_\omega H^1_!(\overline{S}_{K_f^{H,s}}, \mathbf{C}) \cong S_0(K_f^{H,s}, \omega, \mathbf{C})$. The compact open subgroup $K_f^{H,s}$ that we defined for our unramified Hecke character χ after choosing a factorization $\chi = \mu_1/\mu_2$ is given by $\mathrm{GL}_2(\mathcal{O}_v)$ at all places $v \notin S$ for some finite set of places S at which the μ_i are ramified. Note that for each π with $\pi_f^{K_f^{H,s}} \neq 0$ the set Σ_π in Taylor's theorem is a subset of the set comprising the places above p, the places in S and their complex conjugates, and the places ramified in $F/\mathbf{Q}$. Note that if we take the factorization used in Corollary 4.18 then S coincides with the places ramified in $F/\mathbf{Q}$ and hence we have that for each π with $\pi_f^{K_f^{H,s}} \neq 0$ the set Σ_π is contained in the set Σ_0 used in Proposition 7.16.

To apply Taylor's result we need the central character $\omega = \mu_1\mu_2$ of the cuspforms arising in Theorem 6.3 to be cyclotomic. The anticyclotomic characters μ_i used in the definition of the Eisenstein cohomology class will in general not be such that their product is cyclotomic. It is possible, however, to factor $\chi = \eta_1/\eta_2$ with $(\eta_1\eta_2)^c = \eta_1\eta_2$ by the following Lemma (take $\eta_1 = \mu$ and $\eta_2 = (\overline{\mu}^c)^{-1}$):

Lemma 7.24. *Let $\chi : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ be a Hecke character of infinity type $z^{2m}\overline{z}^{2n}$, for $m, n \in \mathbf{Z}$, such that $\chi^c = \overline{\chi}$. Then there exists a Hecke character $\mu : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of infinity type $z^m\overline{z}^n$ such that $\chi = \mu\overline{\mu}^c$. Furthermore, if χ is unramified away from Σ_0 then we can find such a μ ramified only at places in the set Σ_0.*

Proof. We first reduce to the case of finite order characters (i.e. $m = n = 0$). We again use Greenberg's character $\mu_G : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ of infinity type z^{-1} such that $\mu_G^c = \overline{\mu}_G$ and μ_G is ramified exactly at the primes ramified in $F/\mathbf{Q}$ (cf. Lemma 3.18). Given χ as in the Lemma it suffices to prove the Lemma for the character $\chi\mu_G^{2m}\overline{\mu}_G^{2n} = \chi\mu_G^m(\overline{\mu}_G^c)^m\overline{\mu}_G^n(\mu_G^c)^n$, which has trivial infinite component.

For finite order characters we argue as follows: By assumption we have that

$$\chi \equiv 1 \text{ on } \mathrm{Nm}_{F/\mathbf{Q}}(\mathbf{A}_F^*) \subset \mathbf{A}_\mathbf{Q}^* \subset \mathbf{A}_F^*.$$

Thus χ restricted to $\mathbf{Q}^*\backslash\mathbf{A}_\mathbf{Q}^*$ is either the quadratic character of $F/\mathbf{Q}$ or trivial. Since our finite order character has trivial infinite component, χ has to be trivial on $\mathbf{Q}^*\backslash\mathbf{A}_\mathbf{Q}^*$. Hilbert's Theorem 90 then implies that there exists μ such that $\chi = \mu/\mu^c$.

To control the ramification we analyze this last step closer: χ factors through $\mathbf{A}_F^* \to A$, where A is the subset of $\mathbf{A}_F^*$ of elements of the form x/x^c and the map is $x \mapsto x/x^c$. If $y \in A \cap F^*$ then y has trivial norm and so by Hilbert's Theorem 90, $y = x/x^c$ for some $x \in F^*$. Thus the induced character $A \to \mathbf{C}^*$ vanishes on $A \cap F^*$. This implies that there is a continuous finite order character $\mu : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ which restricts to this character on A and $\chi = \mu/\mu^c$ (this argument is taken from the proof of Lemma 1 in [T2]). If χ is unramified, we can similarly conclude that the induced character vanishes on $A \cap \prod_{v \notin \Sigma_0} \mathcal{O}_v^*$ and therefore find μ on $F^* \backslash \mathbf{A}_F^* / \prod_{v \notin \Sigma_0} \mathcal{O}_v^*$ restricting to the character $A \to \mathbf{C}^*$: Writing $U_{F,\ell} = \prod_{v|\ell} \mathcal{O}_v^*$ for a prime ℓ in $\mathbf{Q}$ we have

$$H^1(\mathrm{Gal}(F/\mathbf{Q}), \prod_{\ell \text{ unramified in } F/\mathbf{Q}} U_{F,\ell}) \hookrightarrow \prod_{\ell} H^1(\mathrm{Gal}(F/\mathbf{Q}), U_{F,\ell})$$

and

$$H^1(\mathrm{Gal}(F/\mathbf{Q}), U_{F,\ell}) \cong H^1(G_v, \mathcal{O}_v^*) = 1$$

since $F_v/\mathbf{Q}_\ell$ is unramified. If $y \in A \cap \prod_{v \notin \Sigma_0} \mathcal{O}_v^*$ then y has trivial norm in each $\mathcal{O}_v^*, v \notin \Sigma_0$. By what we just showed this implies that there exists $x \in \prod_{v \notin \Sigma_0} \mathcal{O}_v^* \cap \mathbf{A}_F^*$ such that $y = x/x^c$. The image of y under the induced character therefore equals $\chi(x) = 1$, as claimed above. $\square$

To associate now Galois representations to the cuspforms π congruent to our Eisenstein cohomology class (i.e., with Hecke eigenvalues $a_v(\pi)$ congruent to the eigenvalues of the Eisenstein cohomology class), we twist the forms by η_2/μ_2. Since $\mu_1 = \chi\mu_2$ this gives cuspforms with central character $\eta_1\eta_2$, so we can apply Theorem 7.21. Then we "untwist" the resulting Galois representation $\rho_{\pi \otimes \eta_2/\mu_2}$ by this finite order character to get a Galois representation ρ' with $\mathrm{trace}(\rho'(\mathrm{Frob}_v)) = a_v(\pi)$ for $v \notin \Sigma_0$. We will in the following suppress this twisting process and just denote the end product ρ' by ρ_π.

To apply Urban's result to the cuspforms congruent to the Eisenstein cohomology class we have to check that the eigenvalue at $\mathfrak{p}$ of the latter is a p-adic unit:

Lemma 7.25. *The Hecke eigenvalue $a_{\mathfrak{p}}((\mu_1, \mu_2))$ of Lemma 3.11 lies in $\mathcal{O}_\chi^*$.*

Proof. Denoting the place corresponding to $\mathfrak{p}$ by v_0 we have $a_{\mathfrak{p}}((\mu_1, \mu_2)) = \mu_{1,v_0}^{-1}(\pi_{v_0}) + p\mu_{2,v_0}^{-1}(\pi_{v_0})$. By Lemma 3.21 the first summand has valuation at v_0 equal to 1 (the infinity type of μ_1 is z), the second summand equal to 0, so their sum lies in $\mathcal{O}_\chi^*$. $\square$

7.3.2 Constructing the lattice

By the Eichler-Shimura-Harder isomorphism $\mathbf{T}_\chi$ from Definition 6.1 is isomorphic to the $\mathcal{O}_\chi$-subalgebra $\mathbf{T}$ of $\mathrm{End}_{\mathcal{O}_\chi}(S_0(K_f^{H,s}, \omega, \mathbf{C}))$ generated by the Hecke operators T_v for $v \notin S$, where S is the set of places ramified in $F/\mathbf{Q}$. Recall that $S_0(K_f^{H,s}, \mathbf{C}) \cong \oplus \pi_f^{K_f^{H,s}}$. Consider all the cuspidal automorphic representations with central character $\omega = \mu_1\mu_2$ with $\pi_f^{K_f^{H,s}} \neq 0$ and denote them by $\{\pi_1, \dots, \pi_m\}$. There are finitely many such because $H^1_!(\overline{S}_{K_f^{H,s}}, \mathbf{C})$ has finite dimension.

As explained in the previous section we have associated Galois representations ρ_{π_i} for $i = 1, \dots, m$ with $\mathrm{trace}(\rho_{\pi_i}(\mathrm{Frob}_v)) = a_v(\pi_i)$ and $\det(\rho_{\pi_i}(\mathrm{Frob}_v)) = \omega^{-1}(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v)$ for $v \notin \Sigma_0$, where $\mathfrak{P}_v$ denotes the maximal ideal of $\mathcal{O}_v$ and $\Sigma_0 = S \cup \{w|p\}$.

Let L_i be a finite extension of F_χ such that $\rho_{\pi_i} : G_{\Sigma_0} \to \mathrm{GL}_2(L_i)$, $L := \cup_i L_i$, and $A := \prod_i L_i$. We will now show that we can embed $\mathbf{T}$ in A.

We define the following $\mathcal{O}_\chi$-algebra map:

$$\mathbf{T} \to A: \; T_v \mapsto (a_v(\pi_i))_i.$$

We claim that this map is injective: By definition, $\mathbf{T} \hookrightarrow \oplus_i \mathrm{End}_{\mathcal{O}_\chi}(V_{\pi_i}^{K_f^{H,s}})$, where we denote by V_π the representation space of π. Since T_v acts on π_i by $a_v(\pi_i)$, the image in each summand is given by the $\mathcal{O}_\chi$-algebra generated by the $a_v(\pi_i)$'s. Note that, in fact, $\mathbf{T} \hookrightarrow \prod_i \mathcal{O}_{L_i}$ since $a_v(\pi_i) \in \mathcal{O}_{L_i}$. We can therefore view each $\mathcal{O}_{L_i}$, $i = 1, \dots, m$ and $\prod_i \mathcal{O}_{L_i}$ as a $\mathbf{T}$-module.

For later, we remark that $\mathbf{T} \otimes_{\mathcal{O}_\chi} F_\chi = A$. This follows from $\mathbf{T} \otimes_{\mathcal{O}_\chi} L = \prod_i L$ by comparing dimensions. For the latter, observe that on the one hand, $\dim_L(\mathbf{T} \otimes_{\mathcal{O}_\chi} L)$ is clearly less than or equal to m. On the other hand, each homomorphism $\mathbf{T} \to \mathcal{O}_L$ arising from the projection onto one of the m factors of $\prod_i \mathcal{O}_{L_i} \otimes \mathcal{O}_L$ gives rise to a minimal prime, and these are distinct by (strong) multiplicity one, so the dimension of $\mathbf{T} \otimes_{\mathcal{O}_\chi} L = \prod_{\text{minimal primes } \mathcal{P}} (\mathbf{T}/\mathcal{P}) \otimes_{\mathcal{O}_L} L$ is also bounded below by m.

We also want to remark that $\oplus_{i=1}^m \mathrm{trace}(\rho_{\pi_i})(\sigma) \in \mathbf{T}$ for all $\sigma \in G_{\Sigma_0}$. This follows from the Chebotarev density theorem (which tells us that the Frobenius elements of unramified primes in a Galois extension are dense in the Galois group) and the continuity of the ρ_{π_i} since $\mathbf{T}$ is a finite $\mathcal{O}_\chi$-algebra.

From now on we will assume that $\mathbf{T}/I \neq 0$, where I is the ideal corresponding to $\mathbf{I}_{\mu_1,\mu_2} \subset \mathbf{T}_\chi$ (there is nothing to prove otherwise in Proposition 7.16!). Let $\mathfrak{P}$ be the maximal ideal of $\mathbf{T}$ containing I. We now consider the completions of $\mathbf{T}$ and $\prod_i \mathcal{O}_{L_i}$ at $\mathfrak{P}$.

Lemma 7.26. *If $\mathcal{O}_{L_i}$ is not in the kernel of $\prod_i \mathcal{O}_{L_i} \to (\prod_i \mathcal{O}_{L_i})_\mathfrak{P}$ then $\overline{\rho}_{\pi_i}^{\mathrm{ss}} \cong \overline{\mu}_{1,\mathfrak{p}}^{-1} \oplus \overline{\mu}_{2,\mathfrak{p}}^{-1}\overline{\epsilon}$.*

Proof. A factor $\mathcal{O}_{L_i}$ is not in the kernel of this localization if and only if

$$0 \neq \mathfrak{P}\mathcal{O}_{L_i} \subset \mathfrak{m}_{\mathcal{O}_{L_i}}.$$

Since we have $T_v - \mu_1^{-1}(\mathfrak{P}_v) - \mu_2^{-1}\mathrm{Nm}(\mathfrak{P}_v) \in \mathfrak{P}$ for all $v \notin S$ we must have that $a_v(\pi_i) - \mu_1^{-1}(\mathfrak{P}_v) - \mu_2^{-1}\mathrm{Nm}(\mathfrak{P}_v)$ lies in the maximal ideal of $\mathcal{O}_{L_i}$. We deduce that π_i has Hecke eigenvalues congruent to those of the Eisenstein cohomology class.

By definition we get that the characteristic polynomial of $\overline{\rho}_{\pi_i}(\mathrm{Frob}_v)$ for $v \notin \Sigma_0$ is $x^2 - (\overline{\mu}_{1,\mathfrak{p}}^{-1}(\mathrm{Frob}_v) + \overline{\mu}_{2,\mathfrak{p}}^{-1}\overline{\epsilon}(\mathrm{Frob}_v))x + \overline{\omega}_\mathfrak{p}^{-1}\overline{\epsilon}(\mathrm{Frob}_v)$. By the Chebotarev density theorem any element of G_{Σ_0} can be approximated by such Frobenius elements. Since ρ_{π_i} is continuous we have that $\overline{\rho}_{\pi_i}$ factors through a finite Galois extension and that for any element σ in this extension the characteristic polynomial is given by $x^2 - (\overline{\mu}_{1,\mathfrak{p}}^{-1}(\sigma) + \overline{\mu}_{2,\mathfrak{p}}^{-1}\overline{\epsilon}(\sigma))x + \overline{\omega}_\mathfrak{p}^{-1}\overline{\epsilon}(\sigma)$. But since this agrees with the characteristic polynomial of the representation $\overline{\mu}_{1,\mathfrak{p}}^{-1} \oplus \overline{\mu}_{2,\mathfrak{p}}^{-1}\overline{\epsilon}$ the claim follows from:

Theorem 7.27 (Brauer-Nesbitt). *If ρ_1, ρ_2 are two finite dimensional representations of a finite group G acting on vector spaces V_1, V_2 over a field then*

$$\rho_1^{\mathrm{ss}} \cong \rho_2^{\mathrm{ss}} \Leftrightarrow \textit{characteristic poly}(\rho_1(g)) = \textit{characteristic poly}(\rho_2(g)) \ \textit{for all } g \in G.$$

□

Denote the "surviving" index set by J (which is non-trivial since by assumption $\mathbf{T}_\mathfrak{P}/I \cong \mathbf{T}/I \neq 0$) and write $A_\mathfrak{P} = \prod_{i \in J} L_i \subset A$.

We will now follow the method of [W86] and [W90], with modifications by Skinner in [S02b], to construct the finite $\mathbf{T}_\mathfrak{P}$-modules $\mathcal{L}_1 \subset \mathcal{L}$ in Proposition 7.20. In

Proposition 7.16 we consider two cases: $\rho = \chi_{\mathfrak{p}}\epsilon$ or $\rho = \chi_{\mathfrak{p}}^{-1}\epsilon^{-1}$. In the following we deal with the first case; the modifications necessary for the second being obvious. So from now on $\rho = \chi_{\mathfrak{p}}\epsilon$ and we denote the place $\mathfrak{p}$ at which it has negative Hodge-Tate weight by w. Let $\rho_i = \rho_{\pi_i} \otimes \mu_{1,p}$. By the preceding lemma $\overline{\rho}_i^{\mathrm{ss}} \cong \mathbf{1} \oplus \overline{\rho}$ for $i \in J$.

We consider the $\mathbf{T}_{\mathfrak{P}}$-module $W_{\mathfrak{P}} = A_{\mathfrak{P}} \oplus A_{\mathfrak{P}}$. Fix $\sigma_0 \in I_w$ such that $\rho(\sigma_0) \not\equiv 1 \mod \mathfrak{p}$ (this is possible since the Hodge-Tate weight of ρ at w is -1). We fix a basis of the representations ρ_i for $i \in J$ such that

- $\rho_i(\sigma_0) = \begin{pmatrix} \alpha_i & 0 \\ 0 & \beta_i \end{pmatrix}$ with $\alpha_i, \beta_i \in \mathcal{O}_\chi$, $(\alpha_i), (\beta_i) \in \mathbf{T}_{\mathfrak{P}}$ (and $\alpha_i \not\equiv \beta_i \equiv 1 \mod \mathfrak{p}$),
- $\rho_i : G_{\Sigma_0} \to \mathrm{GL}_2(\mathcal{O}_{L_i})$,
- $\rho_i|_{D_w} = \begin{pmatrix} \Psi_2^{(i)} & 0 \\ * & \Psi_1^{(i)} \end{pmatrix}$ with $\Psi_1^{(i)}$ unramified.

For the first condition we note that by Hensel's Lemma the distinct eigenvalues of $\overline{\rho}_i(\sigma_0)$ lift to distinct eigenvalues of $\rho_i(\sigma_0)$ in $\mathcal{O}_\chi$. That it is possible to find a Galois stable lattice is a standard argument using the compactness of G_{Σ_0} and the continuity of ρ_i. The third condition uses Theorem 7.23 on the ordinarity of ρ_{π_i} and the fact that $\mu_{1,\mathfrak{p}}\epsilon$ is unramified at w.

Now put a Galois action on $W_{\mathfrak{P}}$ via $\tilde{\rho} := \oplus_{i \in J}\rho_i$. The two actions commute and from now on we consider $W_{\mathfrak{P}}$ as a $\mathbf{T}_{\mathfrak{P}}[G_{\Sigma_0}]$-module.

Definition 7.28. A lattice $\mathfrak{L}$ in $W_{\mathfrak{P}}$ is a finitely generated $\mathbf{T}_{\mathfrak{P}}$-module such that $\mathfrak{L} \otimes_{\mathcal{O}_\chi} F_\chi = W_{\mathfrak{P}}$. By a stable lattice we mean a Galois stable lattice.

We first note that $\prod_{i \in J}(\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}) \subset W_{\mathfrak{P}}$ is a stable lattice. We modify it as follows: Write $\tilde{\rho}(\sigma) = \begin{pmatrix} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{pmatrix}$. Since $\oplus_{i \in J}\mathrm{trace}(\rho_i)(\sigma) \in \mathbf{T}_{\mathfrak{P}}$ for any $\sigma \in G_{\Sigma_0}$ we get that $a_\sigma + d_\sigma \in \mathbf{T}_{\mathfrak{P}}$ and $a_\sigma\alpha_i + d_\sigma\beta_i \in \mathbf{T}_{\mathfrak{P}}$. Together with $\alpha_i \not\equiv \beta_i \mod \mathfrak{p}$ we deduce $a_\sigma, d_\sigma \in \mathbf{T}_{\mathfrak{P}}$.

The c_σ lie in $\prod_{i \in J}\mathcal{O}_{L_i}$ by assumption. Because $\mathcal{O}_\chi$ is a discrete valuation ring, any two lattices are commensurable, so there exists an $x \in \mathcal{O}_\chi$ such that $xc_\sigma \in \mathbf{T}_{\mathfrak{P}}$

for all σ. Replacing $\tilde{\rho}$ by $\begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix} \tilde{\rho} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ we can ensure that $\tilde{\rho}$ stabilizes the lattice $\mathcal{L}_0 := \prod_{i \in J} \frac{1}{x} \mathcal{O}_{L_i} \oplus x\mathbf{T}_{\mathfrak{P}}$.

In order to find a lattice $\mathcal{L}$ that satisfies the requirements of the Proposition, we apply the sequence of Lemmas from [W86] and [W90]:

Lemma 7.29. *Suppose $\mathfrak{L}$ is any stable lattice in $W_{\mathfrak{P}}$. Then any irreducible $\mathbf{T}_{\mathfrak{P}}[G_{\Sigma_0}]$-quotient V of $\mathfrak{L}/I$ satisfies either $V \cong \mathbf{T}_{\mathfrak{P}}/\mathfrak{P}$ with trivial G_{Σ_0}-action ("type 1")or with G_{Σ_0}-action via $\overline{\rho}$ ("type ρ").*

Proof. The ρ_{π_i} satisfy that the characteristic polynomials of $\rho_{\pi_i}(\mathrm{Frob}_v)$ are $x^2 - a_v(\pi_i)x + \omega^{-1}(\mathfrak{P}_v)\mathrm{Nm}_{F/\mathbf{Q}}(\mathfrak{P}_v)$. This implies that we have

$$\tilde{\rho}(\mathrm{Frob}_v)^2 - T_v\mu_1(\mathfrak{P}_v)\tilde{\rho}(\mathrm{Frob}_v) + \rho(\mathrm{Frob}_v) = 0 \text{ on } \mathfrak{L} \text{ for } v \notin \Sigma_0.$$

By the definition of I we deduce that $(\mathrm{Frob}_v - 1)(\mathrm{Frob}_v - \rho(\mathrm{Frob}_v))$ annihilates $\mathfrak{L}/I$. The statement follows by another application of the Chebotarev density and Brauer-Nesbitt theorems. □

Definition 7.30. A finite $\mathbf{T}_{\mathfrak{P}}[G_{\Sigma_0}]$-module is said to be of type ρ (resp. type 1) if all irreducible subquotients are of type ρ (resp. type 1).

We seek a stable lattice $\mathfrak{L} \subset \mathfrak{L}_0$ having a filtration

$$0 \to (\text{type } \rho) \to \mathfrak{L}/I \to (\text{ type } 1) \to 0$$

and such that $\mathfrak{L}/I$ has no type ρ quotients.

Lemma 7.31. *Given any stable lattice $\mathfrak{L}$ there exists a stable sublattice $\mathfrak{L}'$ such that $\mathfrak{L}/\mathfrak{L}'$ has type ρ, and if $\mathfrak{L}'' \subset \mathfrak{L}$ is a stable sublattice such that $\mathfrak{L}/\mathfrak{L}''$ has type ρ, then $\mathfrak{L}' \subset \mathfrak{L}''$.*

Proof. This is proved exactly as Proposition 3.2 of [W86]. Set $\mathfrak{L}' = \bigcap\{\mathfrak{L}'' \subset \mathfrak{L} \text{ stable } | \mathfrak{L}/\mathfrak{L}'' \text{ type } \rho\}$. If $\mathfrak{L}'$ is not a lattice then $\mathfrak{L}/\mathfrak{L}' \otimes_{\mathcal{O}_\chi} F_\chi \neq 0$ and any irreducible constituent of $\mathfrak{L}/\mathfrak{L}' \otimes_{\mathcal{O}_\chi} F_\chi$ (as a G_{Σ_0}-module) must be isomorphic to some

ρ_i by the irreducibility of the ρ_{π_i}. Set $(\mathfrak{L}/\mathfrak{L}')^{(1)} = \{x|\sigma_0 x = x\} \subset \mathfrak{L}/\mathfrak{L}'$. By the form of $\rho_i(\sigma_0)$ this is non-zero. But $\mathfrak{L}/\mathfrak{L}' \hookrightarrow \prod_{\mathfrak{L}''} \mathfrak{L}/\mathfrak{L}''$, and each of these is of type ρ. Since $\rho(\sigma_0) \not\equiv 1 \mod \mathfrak{p}$, $(\mathfrak{L}/\mathfrak{L}')^{(1)}$ maps to 0 under this embedding, so we get a contradiction.

□

Note that $\mathfrak{L}'$ in the Lemma has no type ρ quotient by the minimality property ("ρ-deprived sublattice"). Set $\mathfrak{L} = \mathfrak{L}_0'$.

Proposition 7.32 (Proposition 5.4 of [W90]). *Let E be a finite $\mathbf{T}_{\mathfrak{P}}/I[G_{\Sigma_0}]$-module. Suppose that E has no type ρ quotient. Let E_ρ be the maximal type ρ submodule. Then E/E_ρ is of type 1.*

Let $E := \mathfrak{L}/I$ and $E_\rho \subset E$ the maximal type ρ submodule. By the Proposition we deduce that $E_1 := E/E_\rho$ has type 1. Finally we conclude:

Proposition 7.33. *There is an exact sequence of $\mathbf{T}_{\mathfrak{P}}[G_{\Sigma_0}]$-modules*

$$0 \to E_\rho \to E := \mathfrak{L}/I \to E_1 \to 0,$$

where E_ρ is of type ρ, $E_1 \cong \mathbf{T}_{\mathfrak{P}}/I$ is of type 1, and no $\mathbf{T}_{\mathfrak{P}}[G_{\Sigma_0}]$-quotient of E is isomorphic to $\overline{\rho}$.

Proof. It only remains to prove $E_1 \cong \mathbf{T}_{\mathfrak{P}}/I$. Instead of [W86] Lemma 3.4 and [W90] Prop. 5.5. we follow [S02b] in using the element σ_0 to do this. We denote $\mathfrak{L}_0 = \prod_{i\in J} \frac{1}{x}\mathcal{O}_{L_i} \oplus x\mathbf{T}_{\mathfrak{P}} =: \mathfrak{L}_0^{(\rho)} \oplus \mathfrak{L}_0^{(1)}$. Similarly decompose $\mathfrak{L} = \mathfrak{L}^{(\rho)} \oplus \mathfrak{L}^{(1)}$ and $E = E^{(\rho)} \oplus E^{(1)}$.

On $\mathfrak{L}_0^{(\rho)}/I$ the element σ_0 must act either trivially or by $\rho(\sigma)$. Since $\alpha_i \not\equiv 1 \mod \mathfrak{p}$ it acts via ρ. Similarly, σ_0 acts trivially on $\mathfrak{L}_0^{(1)}/I$. Since $\mathfrak{L}_0/\mathfrak{L}$ is of type ρ this implies $\mathfrak{L}_0^{(1)} \subset \mathfrak{L}$. Since clearly $\mathfrak{L}^{(1)} \subset \mathfrak{L}_0^{(1)}$, we have $\mathfrak{L}_0^{(1)} = \mathfrak{L}^{(1)} \cong \mathbf{T}_{\mathfrak{P}}$.

Now $E_1 = E/E_\rho$ has type 1, so $E^{(\rho)} \subset E_\rho$ and $E^{(1)} \twoheadrightarrow E_1$. Since $E_1 = \mathfrak{L}_0^{(1)}/I$ is type 1 we also get $E^{(1)} \hookrightarrow E_1$. This concludes the proof that $E_1 = \mathfrak{L}_0^{(1)}/I \cong \mathbf{T}_{\mathfrak{P}}/I$. □

For Proposition 7.20 we now take $\rho = \chi_{\mathfrak{p}}\epsilon$, $T = \mathbf{T}_{\mathfrak{P}}$, and I the ideal generated by the Eisenstein ideal in $\mathbf{T}_{\mathfrak{P}}$. It is clear from the definitions that the $\mathcal{O}_\chi$-algebra structure map surjects onto $T/I \cong \mathbf{T}_\chi/\mathbf{I}_{\mu_1,\mu_2}$. We rename $\mathcal{L}_1 := E_\rho$ and $\mathcal{L} := E$. The proof of the previous Proposition shows that E_ρ equals $\mathfrak{L}^{(\rho)}/I$. Since $\mathfrak{L}^{(\rho)}$ is a faithful $\mathbf{T}_{\mathfrak{P}}$-module, we obtain $\mathrm{Fitt}_{\mathbf{T}_{\mathfrak{P}}}(\mathfrak{L}^{(\rho)}) = (0)$, from which it follows that $\mathrm{Fitt}_{\mathbf{T}_{\mathfrak{P}}}(E_\rho) \subset I$, as desired. That $0 \to \mathcal{L}_1 \to \mathcal{L} \to \mathcal{L}/\mathcal{L}_1 \to 0$ is split as a sequence of $\mathbf{T}_{\mathfrak{P}}[I_w]$-modules (even $\mathbf{T}_{\mathfrak{P}}[D_w]$-modules!) follows from the form of $\tilde{\rho}|_{D_w}$. Applying Proposition 7.20 this now proves Proposition 7.16.

7.4 Dealing with ramification at places other than w

It is possible to prove a lower bound for the smaller Selmer groups $\mathrm{Sel}(\Sigma', M)$ for non-trivial $\Sigma' \subset \Sigma_0 \backslash \{w|p\}$ after imposing an additional condition (independent of the character χ) on our prime $\mathfrak{p}$:

Proposition 7.34. *Assume in addition that $\ell \not\equiv \pm 1 \mod \mathfrak{p}$ for $\ell \mid d_F$. Then for $\Sigma' = \Sigma_0 \backslash \{w|p\} = \{v|d_F\}$ we have*

$$\mathrm{val}_p(\#\mathrm{Sel}(\Sigma', M)^\vee) \geq \mathrm{val}_p(\#(\mathcal{O}_\chi/(L^{\mathrm{alg}}(0,\chi))))$$

for $M = \mathcal{O}_\chi(\chi_{\mathfrak{p}}\epsilon)$ or $M = \mathcal{O}_\chi((\chi_{\mathfrak{p}}\epsilon)^c) = \mathcal{O}_\chi(\chi_{\mathfrak{p}}^{-1}\epsilon^{-1})$.

Proof. By the definition of $\mathrm{Sel}(\Sigma', M)$ we have to show that the extension $\mathcal{L} = E$ constructed in the previous section is split when viewed as a $\mathbf{T}_{\mathfrak{P}}[I_v]$-extension for $v|d_F$. By construction $\tilde{\rho}$ acts on $\mathcal{L}$ by $\begin{pmatrix} \rho & * \\ 0 & \mathbf{1} \end{pmatrix} \mod I$. Since $\mathbf{T}_{\mathfrak{P}}/I \cong \mathcal{O}_\chi/\mathfrak{p}^n$ for some n as $\mathcal{O}_\chi$-algebras this corresponds to acting by $\begin{pmatrix} \rho & * \\ 0 & \mathbf{1} \end{pmatrix} \in \prod_{i \in J} \mathrm{GL}_2(\mathcal{O}_{L_i}/\mathfrak{p}^n)$. We will show for each ρ_i that, in fact, the inertia groups I_v with $v|d_F$ act trivially. By assumption $\rho(I_v) = 1$, so $\rho_i(I_v) = \{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod \mathfrak{p}^n\}$. Suppose now that $\rho_i|_{I_v} \not\equiv \mathbf{1} \mod \mathfrak{p}^n$. Then there must exist $x \in I_v^{\mathrm{tame}}$ such that $\rho_i(x) \equiv \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$ with $b \notin \mathfrak{p}^n$.

Let $\sigma_v \in G_v$ be any lift of Frob_v. Then $\rho_i(\sigma_v) \equiv \begin{pmatrix} \rho(\sigma_v) & * \\ 0 & 1 \end{pmatrix}$ and so $\rho_i(\sigma_v x \sigma_v^{-1}) = \rho_i(x^{q_v}) \equiv \begin{pmatrix} 1 & bq_v \\ 0 & 1 \end{pmatrix}$ (for $q_v = \#\mathcal{O}_v/\mathfrak{P}_v$) is also congruent to $\begin{pmatrix} 1 & \rho(\sigma_v)b \\ 0 & 1 \end{pmatrix}$. Since ρ is anticyclotomic and v is fixed under complex conjugation we get $\rho(\sigma_v) = \rho(\sigma_v^c) = \rho^{-1}(\sigma_v)$, or $\rho(\sigma_v) = \pm 1$. Under the additional assumption on $\mathfrak{p}$ the congruence for $\rho_i(\sigma_v x \sigma_v^{-1})$ cannot exist, contradicting our assumption of a non-trivial action of I_v.

□

Remark 7.35. Conjecturally, the p-valuations of the two sides in Proposition 7.34 are equal. When $\#\mathrm{Cl}(F) = 1$, this has been proved by very different methods in [Gu93] using the 2-variable Main Conjecture of Iwasawa theory proved by Rubin in [Ru2].

Since the notation in [Gu93] is very different we briefly explain the translation to our setup: For Ψ the Größencharakter attached to an elliptic curve defined over $\mathbf{Q}$ with complex multiplication by $\mathcal{O}$, Guo shows in [Gu93] that for $0 < j < k$ and $p - 1 > k$ the p-valuation of $L^{\mathrm{alg}}(0, \overline{\Psi}^j \Psi^{-k})$ equals the p-valuation of the strict Selmer group of the 1-dimensional Galois representation $(\Psi^k \overline{\Psi}^{-j})_{\mathfrak{p}}$. In the class number one case the only unramified Hecke character of infinity type z^2 is $\chi = \mu_G^{-2}$ (see Lemma 3.18 for the definition of μ_G) and there exists an elliptic curve defined over $\mathbf{Q}$ with complex multiplication by $\mathcal{O}$ and associated Größencharakter μ_G. The proof in [Gu93] extends to $k = j = 1$ for which the Selmer group in [Gu93] agrees with $\mathrm{Sel}(\Sigma', \mathcal{O}_\chi((\chi_{\mathfrak{p}}\epsilon)^{-1}))$ in Proposition 7.34 if $\Psi = \mu_G$. Applying the functional equation and using $\mu_G^c = \overline{\mu}_G$ one can show that the special L-value in [Gu93] has the same p-valuation as $L^{\mathrm{alg}}(0, \chi)$.

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