Modular forms (Lent 2007)
Example sheet 4

1. (a) Let p be a prime and let e be a positive integer. Show that $|\mathrm{SL}_2(\mathbb{Z}/p^e\mathbb{Z})| = p^{3e}(1-1/p^2)$.
   (b) Show that $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(N)] = |\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})| = N^3 \prod_{p|N}(1-1/p^2)$.

2. For an integer h put $q_h = e^{2\pi i\tau/h}$. If $f \in S_k(\Gamma)$, $s \in \mathbf{P}^1(\mathbb{Q})$ a cusp, then $s = \sigma(\infty)$ for some $\sigma \in \mathrm{SL}_2(\mathbb{Z})$ and $f|[\sigma]_k$ has an expansion $F(q_h) = \sum_{n>0} a_n q_h^n$ for $h > 0$ the smallest integer such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in (\sigma^{-1}\Gamma\sigma)_\infty$ (call this the *Fourier expansion of* f at s with respect to σ). Prove that there exists κ such that $|a_n| \leq \kappa n^{k/2}$.

3. Let $\Gamma \subset SL_2(\mathbb{Z})$ be a congruence subgroup containing -1. Let $\Gamma_\infty \subset \Gamma$ be the stabiliser of the cusp ∞. Show that if $k > 2$ is even, then the series

$$E_{\Gamma,k}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \frac{1}{(c\tau + d)^k} \qquad \text{where } \gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

   converges and defines an element of $M_k(\Gamma)$ which is not a cusp form. Identify $E_{k,\Gamma}(\tau)$ in the case $\Gamma = \mathrm{SL}_2(\mathbb{Z})$.

4. Prove that for $k \geq 1$ the theta series $\theta(\tau, k)$ converge absolutely and uniformly on compact subsets.

5. Show that $\Gamma_0(4)$ is generated by $\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\pm\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}$. (Hint: Multiply an element of $\Gamma_0(4)$ on the right alternately by $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 4n & 1 \end{pmatrix}$ for appropriate values of n to reduce the absolute value of the entries in the bottom row until one of them becomes 0.)

6. Let $f \in A_{2k}(\Gamma)$ be a non-zero automorphic form for a congruence subgroup Γ of weight $2k \geq 0$. Prove that

$$\sum_{\tau_x} (\mathrm{ord}_{\tau_x}(f)/e_{\tau_x} - k(1 - 1/e_{\tau_x})) = k(2g-2) + k\nu_\infty,$$

   where ν_∞ is the number of cusps of $X(\Gamma)$ and the sum is over $\tau_x \in \mathbb{H}$ giving a set of representatives for the points in $\Gamma\backslash\mathbb{H}$.

[Comments and corrections on this Example sheet to T.Berger@dpmms.cam.ac.uk]