Modular forms (Lent 2007)
Example sheet 3

1. Find a linear relation between E_4^3, E_{12}, and Δ, and use this to prove Ramanujan's congruence

$$\tau(n) \equiv \sigma_{11}(n) \bmod 691.$$

2. Put $\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{d\tau}$.

   (a) Show that $\theta - \frac{k}{12}E_2$ maps M_k (respectively S_k) to M_{k+2} (respectively S_{k+2}).

   (b) Does $\theta - cE_2$, for suitable c, map $\mathbb{C}E_2$ to M_4?

   (c) Express $\theta\Delta$ and θE_{2k} $(1 \leq k \leq 6)$ in terms of $E_{2\ell}$, Δ.

   (d) Prove that $\tau(n) \equiv n\sigma_5(n) \mod 5$.

3. The Dedekind η-function is defined by

$$\eta(\tau) = e^{2\pi i\tau/24}\prod_{n\geq 1}(1-q^n) \text{ for } \tau \in \mathbb{H}.$$

   Show that it satisfies the identities

$$\eta(\tau+1) = e^{2\pi i/24}\eta(\tau) \text{ and } \eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau),$$

   where we take the branch of square-root which is positive on $\mathbb{R}^+$.

4. Let $f(\tau) = \sum c(n)q^n$ be a normalized Hecke eigenform of weight $2k$ which is not a cuspform. Prove that $f = -\frac{B_{2k}}{4k}E_{2k}$.

5. Fix $k \geq 6$. Let $\mathbb{T} \subset \operatorname{End} S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be the subalgebra of endomorphisms generated over $\mathbb{Z}$ by the Hecke operators $T_{2k}(n)$, $n \geq 1$. Let $S_{2k}(\mathbb{Z}) = S_{2k}(\mathrm{SL}_2(\mathbb{Z})) \cap \mathbb{Z}[[q]]$ denote the submodule of cusp forms with integral Fourier coefficients. If $f \in M_{2k}$ denote by $a_n(f)$ the coefficient of q^n in the q-expansion of f. Show that the map

$$\begin{aligned} S_{2k}(\mathbb{Z}) \times \mathbb{T} &\to \mathbb{Z} \\ (f, T_n) &\mapsto a_1(T_n f) \end{aligned}$$

   gives an isomorphism between $S_{2k}(\mathbb{Z})$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, \mathbb{Z})$, which is an isomorphism of $\mathbb{T}$-modules.

6. (a) Let $G_{2k}(\tau) = \sum c(n)q^n$ be the Fourier expansion of the Eisenstein series G_{2k}. Prove that there are constants $\kappa_1, \kappa_2 > 0$, depending only on k, such that

$$\kappa_1 n^{2k-1} \leq |c(n)| \leq \kappa_2 n^{2k-1} \text{ for all } n \geq 1.$$

   (b) Let $f(\tau) = \sum c(n)q^n$ be a modular form of weight $2k$ which is not a cuspform. Prove that there are constants $\kappa_1, \kappa_2 > 0$, depending on f, such that

$$\kappa_1 n^{2k-1} \leq |c(n)| \leq \kappa_2 n^{2k-1} \text{ for all } n \geq 1.$$

[Comments and corrections on this Example sheet to T.Berger@dpmms.cam.ac.uk]