Modular forms (Lent 2007)
Example sheet 2

1. Let $f \in M_k$ and $g \in M_l$ be modular forms. Show that $lf'g - kfg' \in M_{k+l+2}$.

2. Let $d_k = \dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z}))$. Prove that for any d_k-tuple of complex numbers $(a_0, \ldots, a_{d_k-1})$ there exists exactly one modular form of weight k, having these numbers as first Fourier coefficients.

3. (a) Let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathbf{P}^1(\mathbb{C})$ by linear fractional transformations. Prove that the subgroup that maps the unit disc D onto itself is
$$\mathrm{SU}(1,1) = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} : |a|^2 - |b|^2 = 1\}.$$

   (b) Prove that the group $\mathrm{SU}(1,1)$ is conjugate to $\mathrm{SL}_2(\mathbb{R})$ in $\mathrm{SL}_2(\mathbb{C})$ (Hint: use the Cayley transformation $\mathcal{C}$).

   (c) Show that, under the isomorphism $\mathcal{C} : \mathbb{H} \to D$, the action of the element $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ corresponds to the action of $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ on D.

4. Show that the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{H}$ induces an isomorphism
$$\mathrm{SL}_2(\mathbb{R})/\{\pm I\} \to \mathrm{Aut}(\mathbb{H}) \text{ (biholomorphic automorphisms of } \mathbb{H}).$$

5. Show:

   (a) The group $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$.

   (b) The map
$$\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2,\mathbb{R}) \to \mathbb{H}$$
$$\alpha \mathrm{SO}(2,\mathbb{R}) \mapsto \alpha(i)$$
   is a homeomorphism.

6. Let $\gamma \in \mathrm{SL}_2(\mathbb{R})$ with $\gamma \neq \pm I$. Consider its action on the Riemann sphere $\mathbf{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$.

   (a) If $|\mathrm{tr}(\gamma)| < 2$, show that γ has two fixed points in $\mathbf{P}^1(\mathbb{C})$: one in $\mathbb{H}$ and its complex conjugate. Such an element is called *elliptic*.

   (b) If $|\mathrm{tr}(\gamma)| > 2$, show that γ has two fixed points in $\mathbf{P}^1(\mathbb{R})$ and no other fixed points in $\mathbf{P}^1(\mathbb{C})$. Such an element is called *hyperbolic*.

   (c) If $|\mathrm{tr}(\gamma)| = 2$, show that γ has a single fixed point in $\mathbf{P}^1(\mathbb{R})$ and no other fixed points in $\mathbf{P}^1(\mathbb{C})$. Such an element is called *parabolic*.

   (d) Show that all points of $\mathbb{H}$ fixed by an elliptic element of $\Gamma(1)$ are $\Gamma(1)$-equivalent to i or ρ.

7. Show that the extended upper half plane $\mathbb{H}^*$ is connected.

8. If $f(\tau)$ is a modular function of weight 0 then show that $g(\tau) = f(2\tau) + f(\tau/2) + f(\frac{\tau+1}{2})$ is invariant under $\tau \mapsto \tau + 1$ and $\tau \mapsto -1/\tau$ and hence is also a modular function. Express $j(2\tau) + j(\tau/2) + j(\frac{\tau+1}{2})$ in terms of $j(\tau)$ and deduce a recursive formula for the coefficients of $j(\tau)$.

[Comments and corrections on this Example sheet to T.Berger@dpmms.cam.ac.uk]