Modular forms (Lent 2007)
Example sheet 1

1. Let f be an elliptic function with period lattice Λ. Prove that $\sum_{w\in\mathbb{C}/\Lambda} \mathrm{ord}_w(f)w \in \Lambda$.

2. Let Λ be a lattice in $\mathbb{C}$. The series $\Sigma'_{\omega\in\Lambda}\frac{1}{|\omega|^\sigma}$ is convergent for $\sigma > 2$.

3. Consider $\wp(u) := \wp(u;\Lambda_i)$. Show that $\wp'$ has simple zeros at $\frac{1}{2}, \frac{i}{2}, \frac{1+i}{2}$. Using $\wp(mu; m\Lambda) = m^{-2}\wp(u;\Lambda)$ prove that $\wp(iz) = -\wp(z)$ and deduce that $\wp(\frac{1+i}{2}) = 0$. Show that $\wp(\frac{1}{2}) = -\wp(\frac{i}{2}) \in \mathbb{R}$. By computing some dominant terms show that $\wp(\frac{1}{2}) > 0$. For what m does the complex torus $\mathbb{C}/m\Lambda_i$ correspond to the elliptic curve with equation $y^2 = 4x(x-1)(x+1)$?

4. Let f be an even elliptic function with period lattice Λ. Prove that
$$f(u) = c \prod_{(w\in\mathbb{C}/\Lambda)/\{\pm 1\}} (\wp(u) - \wp(w))^{n_w(f)},$$
where
$$n_w(f) = \begin{cases} \mathrm{ord}_w(f) & \text{if } 2w \notin \Lambda \\ \mathrm{ord}_w(f)/2 & \text{if } 2w \in \Lambda \end{cases}$$
and c is a constant.

5. Use the differential equation satisfied by $\wp(z)$ and the Laurent expansion to show that each $G_{2k}(\Lambda)$ can be written as a polynomial in $G_4(\Lambda)$ and $G_6(\Lambda)$ with positive, rational coefficients. [First obtain the relation $\wp''(z) = 6\wp(z)^2 - g_2/2$.]

6. Show that $\mathrm{Im}(\frac{a\tau+b}{c\tau+d}) = \det\binom{ab}{cd}|c\tau+d|^{-2}\mathrm{Im}(\tau)$ for $\binom{ab}{cd} \in \mathrm{GL}_2(\mathbb{R})$ (where $\mathrm{Im}(\tau)$ is the imaginary part of τ). Deduce that if w_1, w_2 is an oriented basis for a lattice L then $aw_1 + bw_2, cw_1 + dw_2$ for $\binom{ab}{cd} \in \mathrm{GL}_2(\mathbb{Z})$ is oriented if and only if $\binom{ab}{cd} \in \mathrm{SL}_2(\mathbb{Z})$.

7. Determine a point in the fundamental domain $\mathcal{F}$, which is equivalent modulo $\Gamma(1)$ to
(a) $\frac{5i+6}{4i+5}$ (b) $\frac{2}{17} + \frac{8}{17}i$.

8. Prove that any modular form of odd weight is 0. (Hint: $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$.) Prove that any modular form of weight not divisible by 4 vanishes at i. (Hint: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}(i) = i$.) Prove that any modular form of weight not divisible by 6 vanishes at $\omega = (-1+\sqrt{3}i)/2$. (Hint: $\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}(\omega) = \omega$.) Prove that $j(\omega) = 0$ and $j(i) = 1728$. Prove the stronger statements that $j(\tau)$ has a triple zero at ω, and $j(\tau) - 1728$ has a double zero at i.

9. Let $f : \mathbb{H} \to \mathbb{C}$ be a holomorphic modular form without zeros (in $\mathbb{H}$). Prove that f is a constant multiple of a power of the discriminant Δ.

10. Prove that $G_{10} = \frac{5}{11}G_4G_6$ and $G_{14} = \frac{30}{143}G_4^2G_6$.

11. Show that the Riemann sphere can be identified with
$$\mathbf{P}^1(\mathbb{C}) = (\mathbb{C}\times\mathbb{C} - \{(0,0)\})/\mathbb{C}^*.$$
Write $(x_0 : x_1)$ for the equivalence class of (x_0, x_1). Exhibit a complex structure using the open subsets $U_0 = \{(x_0 : x_1) | x_0 \neq 0\}$ and $U_1 = \{(x_0 : x_1) | x_1 \neq 0\}$ and give the transition map.

[Comments and corrections on this Example sheet to T.Berger@dpmms.cam.ac.uk]