DENOMINATORS OF EISENSTEIN COHOMOLOGY CLASSES FOR GL_2 OVER IMAGINARY QUADRATIC FIELDS

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ABSTRACT. We study the arithmetic of Eisenstein cohomology classes (in the sense of G. Harder) for symmetric spaces associated to GL_2 over imaginary quadratic fields. We prove in many cases a lower bound on their denominator in terms of a special L-value of a Hecke character providing evidence for a conjecture of Harder that the denominator is given by this L-value. We also prove under some additional assumptions that the restriction of the classes to the boundary of the Borel-Serre compactification of the spaces is integral. Such classes are interesting for their use in congruences with cuspidal classes to prove connections between the special L-value and the size of the Selmer group of the Hecke character.

1. INTRODUCTION

The relationship between the cohomology of an arithmetic subgroup Γ of a connected reductive algebraic group G and the automorphic spectrum of Γ has been studied extensively. In particular, it is well-known that part of the cohomology can be described by cuspidal automorphic forms. G. Harder initiated a program to describe the entire cohomology in terms of cusp forms and Eisenstein series (together with their residues and derivatives). Using Selberg's and Langlands' theory of Eisenstein series he constructed in [26] a complement to the cuspidal cohomology for the groups GL_2 over number fields. These Eisenstein classes can be described as cohomology classes with nontrivial restriction to the boundary of the Borel-Serre compactification of a symmetric space associated to G.

For arithmetic applications one would like to know if this analytically defined decomposition respects the canonical rational and integral structures on group cohomology. Harder proved for GL_2 that the decomposition is, in fact, rational. By the work of Franke and Schwermer [15] a decomposition of the cohomology of a general reductive group into cuspidal and Eisenstein parts and a rationality result for the groups GL_n are now known. Harder also considered the behavior with respect to the integral structure, in particular the case when this decomposition is rational but not integral, which corresponds to an Eisenstein class with integral restriction to the boundary having a denominator. For a detailed exposition of Harder's program we refer to [27].

We continue this analysis of the arithmetic of Eisenstein cohomology classes in the case of GL_2 over an imaginary quadratic field F. In this case, the associated symmetric space is a 3-dimensional real manifold, and the cohomology in degrees 1 and 2 is the most interesting. We prove a lower bound on the denominator of degree 1 Eisenstein classes in terms of a special L-value of a Hecke character, as

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conjectured by Harder. As an example of the results proven, suppose $m \geq n \geq 0$, let $p > \max\{3, m\}$ be a prime split in F, and $\chi : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ a Hecke character with split conductor coprime to p of infinity type $z^{m+2}\overline{z}^{-n}$ (see Theorem 29 for the complete statement of our result). We construct an Eisenstein cohomology class $\mathrm{Eis}\,\omega_\chi$ (for a coefficient system depending on m and n) that is an eigenvector for the Hecke operators at almost all places such that the p-part of its denominator is divisible by the p-part of $L^{\mathrm{alg}}(0, \chi)$. Here $L^{\mathrm{alg}}(0, \chi)$ is an integral normalization of the special L-value (see Theorem 3). In Proposition 16 we analyze when the restriction of $\mathrm{Eis}\,\omega_\chi$ to the boundary of the Borel-Serre compactification of the symmetric space is integral. In particular, we prove this when $m = n$, $p > m + 1$, and $\chi^c(x) := \chi(\overline{x})$ equals $\overline{\chi}(x)$ for all $x \in \mathbf{A}_F^*$.

Such classes are interesting because of the implications for the Selmer group of the p-adic Galois character associated to χ^{-1}: The situation here should be compared to the classical Eisenstein series of weight 2 for $\Gamma_1(p)$ with a character ϵ used by Ribet in [46]. Its q-expansion is p-integral and the constant term involves an L-value of ϵ. Via the congruence (of q-expansions) of the Eisenstein series with a cuspidal Hecke eigenform Ribet proved the converse to Herbrand's theorem. In our case the symmetric space is not hermitian but one might try to use the integral structure coming from Betti cohomology, as carried out for $\mathrm{GL}_{2/\mathbf{Q}}$ in [29] and [51]. If there exists an integral cohomology class with the same restriction to the boundary as $\mathrm{Eis}\,\omega_\chi$ then our result shows that there exists a congruence modulo $L^{\mathrm{alg}}(0, \chi)$ between $\mathrm{Eis}\,\omega_\chi$, multiplied by its denominator, and a cuspidal cohomology class. Via the Eichler-Shimura-Harder isomorphism and the Galois representations attached to cuspidal automorphic representations by the work of Taylor *et al.* (see [52]) one can then construct elements in the Selmer group of χ^{-1} and obtain a lower bound on its size in terms of $L^{\mathrm{alg}}(0, \chi)$. For this application of the results in this paper in the case of constant coefficients see [3].

Note that for this application only the case $m = n$ is of interest since cuspidal cohomology classes do not exist otherwise. Also, since the interior cohomology for complex coefficients in degrees 1 and 2 are isomorphic, we restrict our study to degree 1. For an analysis of denominators of degree 2 Eisenstein cohomology classes associated to unramified characters see [13].

We give a brief sketch of our proof of the lower bound on the denominator in the special case of constant coefficient systems (corresponding to $m = n = 0$): In this case we can treat split or inert primes $p > 3$. Fix embeddings $F \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ and let $\mathfrak{p}$ be the corresponding prime ideal of F dividing p. Let $G = \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_{2/F})$ and B the Borel subgroup of upper-triangular matrices. For any (sufficiently small) compact open subgroup $K_f \subset G(\mathbf{A}_f)$ let S_{K_f} be the differentiable manifold $G(\mathbf{Q}) \backslash G(\mathbf{A})/K_f K_\infty$, where $K_\infty = U(2)\mathbf{C}^* \subset G(\mathbf{R})$. An Eisenstein cocycle for $H^1(S_{K_f}, \mathbf{C})$ is described by a pair of Hecke characters $\phi = (\phi_1, \phi_2)$ with $\phi_{1,\infty}(z) = z$ and $\phi_{2,\infty}(z) = z^{-1}$ and a choice of a function Ψ_{ϕ_f} in the induced representation

$$V_{\phi_f,\mathbf{C}}^{K_f} = \{\Psi : G(\mathbf{A}_f) \to \mathbf{C} | \Psi(bg) = \phi_f(b)\Psi(g) \forall b \in B(\mathbf{A}_f), \Psi(gk) = \Psi(g) \forall k \in K_f\}.$$

We denote this Eisenstein cocycle by $\mathrm{Eis}(\Psi_{\phi_f})$. In Section 3.2 we will make particular choices for Ψ_{ϕ_f} (and corresponding K_f), the newvector $\Psi_{\phi_f}^{\mathrm{new}}$ and the spherical vector $\Psi_{\phi_f}^0$. We prove that $\Psi_{\phi_f}^{\mathrm{twist}}$, a certain finite twisted sum of $\Psi_{\phi_f}^0$, is a multiple of $\Psi_{\phi_f}^{\mathrm{new}}$ which will allow us to translate between the two. The cohomology class $[\mathrm{Eis}(\Psi_{\phi_f}^0)]$ is by construction an eigenvector for the Hecke operators at almost all places (see Lemma 9) and we prove in Proposition 16 that its restriction to the boundary is integral if $\frac{L^{\mathrm{alg}}(-1,\phi_1/\phi_2)}{L^{\mathrm{alg}}(0,\phi_1/\phi_2)}$ is, and proceed to show this is the case if $(\phi_1/\phi_2)^c = \overline{\phi_1/\phi_2}$.

We know from the work of Harder that the cohomology class $[\mathrm{Eis}(\Psi_{\phi_f})]$ is rational, i.e., it lies already in the cohomology with coefficients in a finite extension of F. Since we are interested in the p-adic properties we study, in fact, its image in $H^1(S_{K_f}, \overline{F}_{\mathfrak{p}})$. The denominator $\delta([\mathrm{Eis}(\Psi_{\phi_f})])$ of the Eisenstein cohomology class is the ideal by which it has to be multiplied to lie inside the image of the cohomology with integral coefficients. We prove that

$$\delta([\mathrm{Eis}(\Psi^0_{\phi_f})]) \subseteq (L^{\mathrm{alg}}(0,\phi_1/\phi_2)).$$

By the functoriality of the evaluation pairing a cocycle represents an integral cohomology class exactly when its pairing against all integral cycles is integral. Explicit generators for the integral homology are not known in our case, but we can obtain the desired lower bound on the denominator by integrating $\mathrm{Eis}(\Psi_{\phi_f})$ against one carefully chosen integral cycle. The (relative) cycle we use is motivated by the classical modular symbol: we integrate along the path

$$\sigma : \mathbf{R}_{>0} \to \mathrm{GL}_2(\mathbf{C})$$
$$t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & t \end{pmatrix},$$

or rather a sum of such paths, one for each connected component of S_{K_f}.

This "toroidal" integral vanishes in general for $\Psi^0_{\phi_f}$ but we show that for $\Psi^{\mathrm{twist}}_{\phi_f}$ the result, up to p-adic units, is

$$\int_\sigma \mathrm{Eis}(\Psi^{\mathrm{twist}}_{\phi_f}) \sim \frac{L(0,\phi_1)L(0,\phi_2^{-1})}{L(0,\phi_1/\phi_2)}.$$

We would like to conclude from this that multiplication by at least $L^{\mathrm{alg}}(0,\phi_1/\phi_2)$ is necessary to make our Eisenstein cohomology class integral. For this we need to control the p-adic properties of the numerator. To achieve this we use results by Hida and Finis on the non-vanishing modulo p of the L-values $L^{\mathrm{alg}}(0,\theta\phi_i^{\pm 1})$ as θ varies in an anticyclotomic $\mathbf{Z}_q$-extension for $q \neq p$. We replace $\mathrm{Eis}(\Psi^{\mathrm{twist}}_{\phi_f})$ by another "twisted" version $\mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f})$ for a finite order character θ of conductor q^r, defined by

$$\mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f})(g) = \sum_{x\in(\mathcal{O}_q/q^r)^*} \theta^{-1}(x)\mathrm{Eis}(\Psi^{\mathrm{twist}}_{\phi_f})(g \begin{pmatrix} 1 & -\frac{x}{q^r} \\ 0 & 1 \end{pmatrix}_q),$$

where $\mathcal{O}_q$ is the ring of integers of the completion of F at q. The sum of paths making up the cycle is also weighted by values of θ. See Section 4.1 for the definition of this cycle σ_θ. Up to units the result of this toroidal integral is

$$\int_{\sigma_\theta} \mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f}) \sim \frac{L(0,\phi_1\theta)L(0,\phi_2^{-1}\theta^{-1})}{L(0,\phi_1/\phi_2)}.$$

The results of Hida and Finis allow us (under certain conditions on the conductors of the ϕ_i) to find a character θ such that the numerator is a p-adic unit. Apart from differences in the conditions on the conductors Hida deals only with split p, whilst Finis also treats inert p for constant coefficients. Given a character χ satisfying certain assumptions we prove in Theorem 29 the existence of characters ϕ_1 and

ϕ_2 with $\chi = \phi_1/\phi_2$ for which the L-values in the numerator can be simultaneously controlled. This involves the construction of characters with prescribed ramification and a careful analysis of Artin roots numbers.

The twisting by θ also has the effect of making $\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})$ vanish at the 0- and ∞-cusps of each connected component. By a result of Borel (see Proposition 6) it therefore represents a relative cohomology class with respect to these boundary components. We prove that this relative cohomology class is again rational and that its denominator bounds that of $\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})$ from below. We can therefore interpret the toroidal integral as an evaluation pairing between relative cohomology and homology and deduce that the ideal generated by $L^{\mathrm{alg}}(0, \phi_1/\phi_2)$ gives a lower bound on the denominator of the relative cohomology class represented by $\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})$. We conclude the desired bound on the denominator of $[\mathrm{Eis}(\Psi_{\phi_f}^0)]$ by using the divisibilities

$$\delta([\mathrm{Eis}(\Psi_{\phi_f}^0)]) \subseteq \delta([\mathrm{Eis}(\Psi_{\phi_f}^{\mathrm{twist}})]) \subseteq \delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]) \subseteq \delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}}).$$

Our results generalize and extend the work in [40] for $F = \mathbf{Q}(i)$ and unramified ϕ_1/ϕ_2, where the toroidal integral is calculated for the spherical vector. König proceeds to show in his case that the L-value gives an upper bound on the denominator. Before this, Eisenstein cohomology for imaginary quadratic fields had been studied in [24], [25], and [58]. Previous work on calculating or bounding denominators for GL_2 over $\mathbf{Q}$ and totally real fields include [22], [35], [43], [45], [51], and [57]. [35] and [51] also use twisting techniques and a result by Washington on the non-vanishing modulo p of Dirichlet L-values in cyclotomic towers. New about our method for getting a lower bound is that we introduce the auxiliary cocycle $\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})$ and prove that it represents a relative cohomology class, which allows us to work just with the toroidal integral, making the calculation of additional boundary integrals as in [22], [35] unnecessary. Our method does not allow to prove upper bounds because of the transition to the finite twisted sum, but one might be able to get an upper bound by applying this idea to prove a lower bound on the denominator of the dual cohomology class in degree 2. In principle, our method should extend to general CM-fields, where Hida's result is still applicable. Since the arithmetically interesting classes appear in the middle degrees this would, however, be notationally more cumbersome (but see [43]).

These results generalize part of my thesis [2] under C. Skinner at the University of Michigan, where this problem was considered in the case of constant coefficient systems and split p. The author would like to thank Thanasis Bouganis, Vladimir Dokchitser, Günter Harder, Joachim Schwermer, and Chris Skinner for helpful discussions and an anonymous referee for improvements to the introduction and corrections in the statement of Theorem 3. This article was written during visits to the Max Planck Institute in Bonn and the Erwin Schrödinger Institute in Vienna. The author would like to thank both for their hospitality and support.

2. Notation and Definitions

2.1. **Basic notation.** Let F be an imaginary quadratic field, σ its nontrivial automorphism, $\mathcal{D}$ the different of F, and $d_F = \mathrm{Nm}(\mathcal{D})$ the absolute discriminant. For a place v of F let F_v be the completion of F at v. We write $\mathcal{O}$ for the ring of integers of F, $\mathcal{O}_v$ for the closure of $\mathcal{O}$ in F_v, $\mathfrak{P}_v$ for the maximal ideal of $\mathcal{O}_v$, π_v for a uniformizer of F_v, and $\hat{\mathcal{O}}$ for $\prod_{v \text{ finite}} \mathcal{O}_v$. Complex conjugation is denoted by

$z \mapsto \overline{z}$. We use the notations $\mathbf{A}, \mathbf{A}_f$ and $\mathbf{A}_F, \mathbf{A}_{F,f}$ for the adeles and finite adeles of $\mathbf{Q}$ and F, respectively, and write $\mathbf{A}^*$ and $\mathbf{A}_F^*$ for the group of ideles. Let $p > 3$ be a prime of $\mathbf{Z}$ that does not ramify in F. Fix embeddings $F \hookrightarrow \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p \hookrightarrow \mathbf{C}$ and let $\mathfrak{p}$ be the corresponding prime ideal of F over p.

2.2. **The algebraic group and symmetric spaces.** For any algebraic group $H/\mathbf{Q}$ and any ring A containing $\mathbf{Q}$ we write $H(A)$ for the group of A-valued points. We shall abbreviate $H_\infty = H(\mathbf{R})$. We consider the algebraic group

$$G := \mathrm{Res}_{F/\mathbf{Q}}(\mathrm{GL}_{2/F}).$$

The group $G_0/F = \mathrm{GL}_{2/F}$ contains the Borel subgroup of upper triangular matrices B_0, its unipotent radical U_0, the maximal split torus T_0, and the center Z_0. The restriction of scalars gives corresponding subgroups $B/\mathbf{Q}, T/\mathbf{Q}, U/\mathbf{Q}$ and $Z/\mathbf{Q}$ of G. We single out the element $w_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G(\mathbf{Q})$.

The positive simple root defines a homomorphism

$$\alpha_0 : B_0/F \to \mathbf{G}_m/F$$
$$\begin{pmatrix} t_1 & * \\ 0 & t_2 \end{pmatrix} \mapsto t_1/t_2$$

and we denote by α the corresponding homomorphism $B/\mathbf{Q} \to \mathrm{Res}_{F/\mathbf{Q}}\mathbf{G}_m$. From [26] we take the notation $|\alpha|$ for $|\,| \circ \alpha_{\mathbf{A}} : B(\mathbf{A}) \to \mathbf{C}^*$, where $|\,| : F^* \backslash \mathbf{A}_F^* \to \mathbf{C}^*$ is the idelic absolute value $x \mapsto |x| = \prod_v |x_v|_v$. Here we take the usual normalized absolute values for the local absolute values, in particular, $|x_\infty|_\infty = x_\infty \overline{x}_\infty$ at the complex place.

In $G_\infty = \mathrm{GL}_2(\mathbf{C})$ we choose the subgroup $K_\infty = U(2) \cdot Z_0(\mathbf{C}) = U(2) \cdot \mathbf{C}^*$ containing the maximal compact subgroup of unitary matrices. The symmetric space $X = G_\infty/K_\infty$ can be identified with three-dimensional hyperbolic space $\mathbf{H}_3 = \mathbf{R}_{>0} \times \mathbf{C}$.

The Lie algebra $\mathfrak{g} = \mathrm{Lie}(G/\mathbf{Q})$ is a $\mathbf{Q}$-vector space and we define $\mathfrak{g}_\infty = \mathfrak{g} \otimes_{\mathbf{Q}} \mathbf{R}$. It carries a positive semidefinite K_∞-invariant form, the Killing form

$$\langle X, Y \rangle = \frac{1}{16} \mathrm{trace}(\mathrm{ad}X \cdot \mathrm{ad}Y),$$

and with respect to this form we have an orthogonal decomposition $\mathfrak{g}_\infty = \mathfrak{k}_\infty \oplus \mathfrak{p}$, where $\mathfrak{k}_\infty = \mathrm{Lie}(K_\infty)$ and

$$\mathfrak{p} = \mathbf{R}H \oplus \mathbf{R}E_1 \oplus \mathbf{R}E_2 := \mathbf{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \mathbf{R}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

Put

$$S_\pm := 1/2 \left(\pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes_{\mathbf{R}} 1 - \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \otimes_{\mathbf{R}} i \right) \in \mathfrak{p}_{\mathbf{C}}.$$

A maximal open compact subgroup of $G(\mathbf{A}_f)$ is given by

$$\mathrm{GL}_2(\widehat{\mathcal{O}}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \widehat{\mathcal{O}}, ad - bd \in \widehat{\mathcal{O}}^* \right\}.$$

We will deal with the following congruence subgroups: For an ideal $\mathfrak{N}$ in $\mathcal{O}$ and a finite place v of F put $\mathfrak{N}_v = \mathfrak{N}\mathcal{O}_v$. We define

$$K^1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}), a - 1, c \equiv 0 \mod \mathfrak{N} \right\},$$

$$K^1(\mathfrak{N}_v)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{GL}_2(\mathcal{O}_v), a-1, c\equiv 0 \mod \mathfrak{N}_v\right\},$$

and

$$U^1(\mathfrak{N}_v)=\{k\in \mathrm{GL}_2(\mathcal{O}_v) : \det(k)\equiv 1 \mod \mathfrak{N}_v\}.$$

For any compact open subgroup $K_f \subset G(\mathbf{A}_f)$ the adelic symmetric space

$$S_{K_f} := G(\mathbf{Q})\backslash G(\mathbf{A})/K_\infty K_f$$

has several connected components. In fact, strong approximation implies that the fibers of the determinant map

$$S_{K_f} \twoheadrightarrow \pi_0(K_f) := \mathbf{A}^*_{F,f}/\det(K_f)F^*$$

are connected. Any $\gamma\in G(\mathbf{A}_f)$ gives rise to an injection $j_\gamma : G_\infty \to G(\mathbf{A})$ with $j_\gamma(g_\infty) := (g_\infty,\gamma)$ and, after taking quotients, to a component

$$\Gamma_\gamma\backslash G_\infty/K_\infty \to S_{K_f},$$

where $\Gamma_\gamma := G(\mathbf{Q})\cap \gamma K_f\gamma^{-1}$. This component is the fiber over $\det(\gamma)$. Choosing a system of representatives for $\pi_0(K_f)$ we therefore have

$$S_{K_f}\cong \coprod_{[\det(\gamma)]\in\pi_0(K_f)} \Gamma_\gamma\backslash\mathbf{H}_3.$$

We denote the Borel-Serre compactifications of S_{K_f} and $\Gamma_\gamma\backslash\mathbf{H}_3$ by $\overline{S}_{K_f}$ and $\Gamma_\gamma\backslash\overline{\mathbf{H}}_3$, respectively. Following [5] we write $e(P)=\mathbf{H}_3/A_P\cong U_P(\mathbf{R})$ for each rational Borel subgroup P of G. Here U_P denotes its unipotent radical and A_P the identity component of $P(\mathbf{R})/U_P(\mathbf{R})$, and the action of A_P on $\mathbf{H}_3$ is the geodesic action. The boundary of $\Gamma_\gamma\backslash\overline{\mathbf{H}}_3$ is the union of tori $\Gamma_{\gamma,P}\backslash e(P) =: e'(P)$ with $\Gamma_{\gamma,P}=\Gamma_\gamma\cap P(\mathbf{Q})$ over a set of representatives for the Γ_γ-conjugacy classes of Borel subgroups (equivalently of $B(\mathbf{Q})\backslash G(\mathbf{Q})/\Gamma_\gamma\cong \mathbf{P}^1(F)/\Gamma_\gamma$). We recall from [26] §2.1 and [25] p. 110 that $\partial\overline{S}_{K_f}$ is homotopy equivalent to

$$\partial\tilde{S}_{K_f} := B(\mathbf{Q})\backslash G(\mathbf{A})/K_fK_\infty \cong \coprod_{[\det(\gamma)]\in\pi_0(K_f)}\ \coprod_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\gamma} \Gamma_{\gamma,B^\eta}\backslash\mathbf{H}_3, \tag{1}$$

where $B^\eta(\mathbf{Q})=\eta^{-1}B(\mathbf{Q})\eta$ for $\eta\in G(\mathbf{Q})$ and the boundary component $\Gamma_{\gamma,B^\eta}\backslash\mathbf{H}_3$ gets embedded in $\partial\tilde{S}_{K_f}$ via $g_\infty\mapsto j_{\eta,\gamma}(g_\infty) := \eta(g_\infty,\gamma)$.

2.3. **Hecke characters.** A Hecke character of F is a continuous group homomorphism $\lambda : F^*\backslash\mathbf{A}_F^*\to\mathbf{C}^*$. Such a character corresponds uniquely to a character on ideals prime to the conductor (see [31] §8.2), which we will also denote by λ. The archimedean part $\lambda_\infty : \mathbf{C}^*\to\mathbf{C}^*$ is of the form $z\mapsto \frac{z^a\overline{z}^b}{(z\overline{z})^t}$ for $t\in\mathbf{C}, a,b\in\mathbf{Z}$. We will say that λ has *infinity type* $\frac{z^a\overline{z}^b}{(z\overline{z})^t}$. We define the (incomplete) L-series $L(s,\lambda)$ for $\mathrm{Re}(s)\gg 0$ by the Euler product

$$L(s,\lambda) := \prod_{v\nmid\mathfrak{f}_\lambda}(1-\lambda(\mathfrak{P}_v)\mathrm{Nm}(\mathfrak{P}_v)^{-s})^{-1},$$

where $\mathfrak{f}_\lambda$ is the conductor of λ. This can be continued to a meromorphic function on the whole complex plane and satisfies a functional equation (see e.g., [31] §8.6 or [11] 37).

Define the character λ^c by $\lambda^c(x)=\lambda(\sigma(x))$. Since σ just permutes the Euler factors we have $L(s,\lambda)=L(s,\lambda^c)$. Also let $\lambda^*(x) := \lambda(\sigma(x))^{-1}|x|$.

Recall from [11] p.91 and [41] XIV Theorem 14 the definition of the global root number $W(\lambda)$ appearing in the functional equation. Note that $W(\lambda) = W(\tilde{\lambda})$ for $\tilde{\lambda}$ the associated unitary character $\lambda/|\lambda|$. If $\lambda^* = \lambda$ then one shows using the functional equation that $W(\lambda) = \pm 1$. For λ of infinity type $\frac{z^m}{(z\overline{z})^{m/2}}$ with $m \in \mathbf{Z}$ we have

$$W(\lambda) = i^{-m}(\mathrm{Nm}(\mathfrak{f}_\lambda))^{-1/2} \prod_{v|\mathfrak{f}_\lambda} \tau_v(\lambda) \prod_{v\nmid\mathfrak{f}_\lambda} \lambda(\mathcal{D}_v^{-1}),$$

where the Gauss sum τ_v is given by

$$\tau_v(\lambda_v) = \sum_{\epsilon \in \mathcal{O}_v^*/(1+f_{\lambda,v})} (\lambda \mathbf{e}_F)(\epsilon \pi^{-\mathrm{ord}_v(\mathfrak{f}_\lambda \mathcal{D})}).$$

Here $\mathbf{e}_F$ is the standard additive character of $F\backslash\mathbf{A}_F$ defined by $e_F = e_{\mathbf{Q}} \circ \mathrm{Tr}_{F/\mathbf{Q}}$ in terms of the standard additive character $e_{\mathbf{Q}}$ of $\mathbf{Q}\backslash\mathbf{A}$ normalized by $e_{\mathbf{Q}}(x_\infty) = e^{2\pi i x_\infty}$. Put $\tau(\lambda) = \prod_{v|\mathfrak{f}_\lambda} \tau_v(\lambda)$.

We will use the following formula of Weil as stated in [1] Proposition 2.4:

Proposition 1. *Suppose that λ_1 and λ_2 are unitary Hecke characters of infinity types (k_1, j_1) and (k_2, j_2) with relatively primes conductors $\mathfrak{f}_1$ and $\mathfrak{f}_2$. Then*

$$W(\lambda_1)W(\lambda_2)\lambda_1(\mathfrak{f}_2)\lambda_2(\mathfrak{f}_1) = \begin{cases} W(\lambda_1\lambda_2) & \text{if } (k_1 - j_1)(k_2 - j_2) \geq 0, \\ (-1)^\nu W(\lambda_1\lambda_2) & \text{if } (k_1 - j_1)(k_2 - j_2) < 0, \end{cases}$$

where $\nu = \min\{|k_1 - j_1|, |k_2 - j_2|\}$. □

For ease of reference we record the following:

Lemma 2. *For $\lambda : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ with infinity type $z^a\overline{z}^b$ with $a, b \in \mathbf{Z}$ we denote by $\mathcal{O}_\lambda$ the ring of integers in the finite extension of $F_\mathfrak{p}$ obtained by adjoining the values of the finite part of λ. Then for any $x \in \mathbf{A}_{F,f}^*$*

$$\mathrm{ord}_\mathfrak{p}(\lambda(x)) = -a \cdot \mathrm{ord}_\mathfrak{p}(x_\mathfrak{p}) - b \cdot \mathrm{ord}_{\overline{\mathfrak{p}}}(x_{\overline{\mathfrak{p}}}).$$

Proof. Let v be any finite place of F. Since λ has finite order on $\mathcal{O}_v^*$ it suffices to prove the statement for $\lambda(\pi_v)$ for any uniformizer π_v. If h is the class number of F, we have $\mathfrak{P}_v^h = (\alpha)$ for $\alpha \in \mathcal{O}$ and $\alpha \in \mathcal{O}_w^*$ for $w \neq v$. Now

$$1 = \lambda((\alpha, \alpha, \ldots)) = \lambda_\infty(\alpha)\lambda_v(\alpha) \prod_{w \neq v} \lambda_w(\alpha).$$

Since $\prod_{w\neq v} \lambda_w(\alpha) \in \mathcal{O}_\lambda^*$ we deduce that

$$h \cdot \mathrm{ord}_\mathfrak{p}(\lambda(\pi_v)) = \mathrm{ord}_\mathfrak{p}(\lambda_v(\alpha)) = -\mathrm{ord}_\mathfrak{p}(\lambda_\infty(\alpha)).$$

□

Define $\Omega \in \mathbf{C}$ to be the complex period of a Néron differential ω of an elliptic curve E defined over some number field such that E has complex multiplication by $\mathcal{O}$, E has good reduction at the place above p and $\overline{\omega}$ is a non-vanishing invariant differential on the reduced curve $\overline{E}$.

Let λ be a Hecke character of infinity type $z^a\overline{z}^b$ with $a, b \in \mathbf{Z}$. Precisely for $a > 0$ and $b \leq 0$ or $a \leq 0$ and $b > 0$ the L-value $L(0, \lambda)$ is critical in the sense of Deligne. Damerell showed in this case that $\pi^{\max(-a,-b)}\Omega^{-|a-b|}L(0, \lambda)$ is an algebraic number in $\mathbf{C}$. We recall the following results (due to, amongst others, Shimura, Coates-Wiles, Katz, Hida, Tilouine, de Shalit, and Rubin) about the integrality of the special L-value at $s=0$:

Theorem 3. *Let λ a Hecke character of infinity type $z^a\overline{z}^b$ with conductor prime to p. Assume $a,b \in \mathbf{Z}$ and $a>0$ and $b \leq 0$. Put*

$$L^{\mathrm{alg}}(0,\lambda) := \Omega^{b-a}\left(\frac{2\pi}{\sqrt{d_F}}\right)^{-b}\Gamma(a)\cdot L(0,\lambda).$$

(a) *If p is split then*

$$(1-\lambda(\overline{\mathfrak{p}}))(1-\lambda^*(\overline{\mathfrak{p}}))\cdot L^{\mathrm{alg}}(0,\lambda)$$

lies in the ring of integers of a finite extension of $F_{\mathfrak{p}}$.

(b) *If p is inert and $a>0, b=0$ then for any ideal $\mathfrak{b}$ coprime to $6p$ and the conductor of λ*

$$(\mathrm{Nm}(\mathfrak{b})-\lambda^{-1}(\mathfrak{b}))\cdot L^{\mathrm{alg}}(0,\lambda)$$

lies in the ring of integers of a finite extension of $F_{\mathfrak{p}}$.

References. If p is split then the normalization in (a) is the one appearing in the p-adic L-function constructed by Manin-Vishik, Katz, and others. Together, [36] Chapters 4 and 8, [38] Theorem 5.3.0, and [34] Theorem II prove that it is a p-adic integer in $\widehat{\overline{F}}_{\mathfrak{p}}$. With our fixed embedding $\overline{F} \hookrightarrow \overline{F}_{\mathfrak{p}}$ this shows that the value lies in a finite extension of $F_{\mathfrak{p}}$ and is p-integral. See also [32] Theorem 1.1 and [11] Theorem II.4.14 and II.6.7.

Part (b) uses the relation of elliptic units to special values of L-functions. For the proof in the case when λ is the power of a Grössencharacter of a CM elliptic curve and F has class number one see, for example [49] §7, in particular, Theorem 7.22. To extend to the general case use the arguments in [11] Chapter II. □

Remark.
(1) If p is split then Lemma 2 shows that for $a \geq 2$ the factor $(1-\lambda^*(\overline{\mathfrak{p}}))$ is a p-unit.
(2) If F has class number one, $p>a$, and λ is the power of a Grössencharacter of a CM elliptic curve then [12] Lemma 3.4.5 proves that there always exists an ideal $\mathfrak{b}$ such that $\mathrm{Nm}(\mathfrak{b})-\lambda^{-1}(\mathfrak{b})$ is prime to p.
(3) For completeness we want to mention that for inert primes p additional divisibilities have been obtained in [37], [39], [48], [17], and [9].

2.4. **Modules and Sheaves.** The group $\mathrm{GL}_2(F)$ acts on the F-vector space $M^n := \mathrm{Sym}^n(F^2)$ of homogeneous polynomials of degree n in two variables X and Y with coefficients in F by right translation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.X^iY^{n-i} = (aX+cY)^i(bX+dY)^{n-i}.$$

Applying first the field automorphism σ to the entries a,b,c and d, we get another representation $\overline{M}^n$. We also have one-dimensional representations $F[k,\ell]$ for $(k,\ell)\in\mathbf{Z}^2$, on which $g\in G$ acts by multiplication by $\det^k(g)\cdot\sigma(\det(g))^\ell$. We obtain the representations $M(m,n,k,\ell) := M^m\otimes_F \overline{M}^n \otimes_F F[k,\ell]$. Let $M(m,n,k,\ell)^\vee := \mathrm{Hom}_F(M(m,n,k,\ell),F)$. There is an isomorphism of $\mathrm{GL}_2(F)$-modules

$$M(m,n,k,\ell)^\vee \cong M(m,n,-m-k,-n-\ell)$$

induced by the pairing

$$\langle\, ,\, \rangle : M(m,n,k,\ell) \times M(m,n,-m-k,-n-\ell) \to F,$$

$$X^j Y^{m-j} \overline{X}^k \overline{Y}^{n-k} \times X^\mu Y^{m-\mu} \overline{X}^\nu \overline{Y}^{n-\nu} \mapsto (-1)^{j+k} \binom{m}{j}^{-1} \binom{n}{k}^{-1} \delta_{j,m-\mu}\delta_{k,n-\nu}.$$

This is the coordinatized version of the pairing induced by the determinant pairing on F^2 (cf. [31] p. 169).

For an $\mathcal{O}$-module N we denote $N \otimes_{\mathcal{O}} A$ by N_A for any $\mathcal{O}$-algebra A. Denote by $M(m,n,k,\ell)_{\mathcal{O}}$ the polynomials with $\mathcal{O}$-coefficients. Note that $M(m,n,k,\ell)^\vee_{\mathcal{O}} := \mathrm{Hom}_{\mathcal{O}}(M(m,n,k,\ell), \mathcal{O})$ corresponds under the duality above to

$$\left\{ \sum_{\mu,\nu} a_{\mu,\nu} \binom{m}{\mu} \binom{n}{\nu} X^\mu Y^{m-\mu} \overline{X}^\nu \overline{Y}^{n-\nu} | a_{\mu,\nu} \in \mathcal{O} \right\} \subset M(m,n,k,\ell).$$

We now define local coefficient systems on the symmetric spaces. For $\Gamma \subset G(\mathbf{Q})$ an arithmetic subgroup and N an $\mathcal{O}[\Gamma]$-module we define a sheaf of $\mathcal{O}$-modules on $\Gamma\backslash\mathbf{H}_3$ by

$$\begin{aligned}\widetilde{N}(U) := \{ f : \pi_\Gamma^{-1}(U) \to N \text{ locally constant } : \\ f(\beta x) = \beta.f(x) \forall x \in \pi_\Gamma^{-1}(U) \text{ and } \beta \in \Gamma\},\end{aligned}$$

where $\pi_\Gamma : \mathbf{H}_3 \to \Gamma\backslash\mathbf{H}_3$ is the canonical projection.

Let $K_f \subset G(\mathbf{A}_f)$ be a compact open subgroup and M an $F[G(\mathbf{Q})]$-module. Assume that there exists an $\mathcal{O}$-lattice $M_{\mathcal{O}}$ in M such that $M_{\hat{\mathcal{O}}} = M_{\mathcal{O}} \otimes \hat{\mathcal{O}}$ is stable under K_f. (For $M = M(m,n,k,\ell)$ and $K_f \subset \mathrm{GL}_2(\hat{\mathcal{O}})$ one can take $M_{\mathcal{O}} = M(m,n,k,\ell)_{\mathcal{O}}$.) For each open subset $U \subset S_{K_f}$ we let

$$\widetilde{M_{\mathcal{O}}}(U) := \left\{ f : \pi^{-1}(U) \to M \text{ locally constant } \left| \begin{array}{l} f(\beta g) = \beta.f(g), f(g) \in g_f M_{\hat{\mathcal{O}}} \\ \forall g \in \pi^{-1}(U) \text{ and } \beta \in G(\mathbf{Q}) \end{array} \right. \right\},$$

where $\pi : G(\mathbf{A})/K_\infty K_f \to S_{K_f}$ is the projection. This defines a sheaf of $\mathcal{O}$-modules on S_{K_f} (cf. [55] §1.4, [40] §1.5, and [13] §1.2). For any $\mathcal{O}$-algebra R we define $\widetilde{M}_R$ as $\widetilde{M}_{\mathcal{O}} \otimes \underline{R}$, where $\underline{R}$ is the constant sheaf associated to R.

For $\gamma \in G(\mathbf{A}_f)$ let $M_\gamma := M \cap \gamma.M_{\hat{\mathcal{O}}}$. Then M_γ is a locally free, finitely generated $\mathcal{O}$-module with an action by $\Gamma_\gamma = G(\mathbf{Q}) \cap \gamma K_f \gamma^{-1}$. The two constructions of $\widetilde{M}_{\mathcal{O}}$ and $\widetilde{M}_\gamma$ are compatible with j_γ; one checks that $j_\gamma^*(\widetilde{M}_{\mathcal{O}}) \cong \widetilde{M}_\gamma$.

2.5. **Cohomology.** For a sheaf $\mathcal{F}$ on a topological space X, we denote by $H^i(X,\mathcal{F})$ (resp. $H^i_c(X,\mathcal{F})$) the i-th cohomology group of $\mathcal{F}$ (resp. with compact support), and the interior cohomology, i.e., the image of $H^i_c(X,\mathcal{F})$ in $H^i(X,\mathcal{F})$, by $H^i_!(X,\mathcal{F})$.

Let M be an $F[G(\mathbf{Q})]$-module with $M_{\mathcal{O}} \subset M$ an $\mathcal{O}$-lattice as above and R an $\mathcal{O}$-algebra. Since $S_{K_f} \overset{i}{\hookrightarrow} \overline{S}_{K_f}$ is a homotopy equivalence, we have a canonical isomorphism

$$H^i(S_{K_f}, \widetilde{M}_R) \cong H^i(\overline{S}_{K_f}, i_*\widetilde{M}_R)$$

and in what follows we will replace $i_*\widetilde{M}_R$ by $\widetilde{M}_R$ and also write $\widetilde{M}_R$ for the sheaf $j^* i_* \widetilde{M}_R$ on $\partial\overline{S}_{K_f}$, for $j : \partial\overline{S}_{K_f} \hookrightarrow \overline{S}_{K_f}$.

The decomposition of the adelic symmetric space into connected components gives rise to canonical isomorphisms (see [40] §1.6 and [13] §1.2)

$$H^i(S_{K_f}, \widetilde{M}_R) \cong \bigoplus_{[\det(\gamma)] \in \pi_0(K_f)} H^i(\Gamma_\gamma \backslash \mathbf{H}_3, \widetilde{M_\gamma} \otimes \underline{R})$$

and

$$H^i(\partial \tilde{S}_{K_f}, \widetilde{M}_R) \cong \bigoplus_{[\det(\gamma)] \in \pi_0(K_f)} \bigoplus_{[\eta] \in \mathbf{P}^1(F)/\Gamma_\gamma} H^i(\Gamma_{\gamma, B^\eta} \backslash \mathbf{H}_3, \widetilde{M_\gamma} \otimes \underline{R}).$$

The above cohomology groups and isomorphisms are all functorial in R.

For an arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$ and an $\mathcal{O}[\Gamma]$-module N we can in many cases relate the sheaf cohomology $H^i(\Gamma \backslash \mathbf{H}_3, \widetilde{N}_R)$ to group cohomology $H^i(\Gamma, N_R)$ (for the proof see, e.g., [23]):

Proposition 4. *For $\mathcal{O}$-algebras R in which the orders of all finite subgroups of Γ are invertible there is a natural R-functorial isomorphism*

$$H^i(\Gamma \backslash \mathbf{H}_3, \widetilde{N}_R) \cong H^i(\Gamma, N_R).$$

□

The lemma in [13] §1.1 shows that for any $\mathcal{O}$-algebra R, $R \otimes_{\mathcal{O}} \mathcal{O}[\frac{1}{6}]$ satisfies the conditions of the proposition for any arithmetic subgroup $\Gamma \subset G(\mathbf{Q})$.

For complex coefficient systems we have analytic tools available. For a C^∞-manifold X (like S_{K_f}, $\partial \tilde{S}_{K_f}$, or $\Gamma \backslash \mathbf{H}_3$) denote by $\Omega^i(X)$ the space of $\mathbf{C}$-valued C^∞-differential i-forms and by $\Omega^i(X, M_\mathbf{C}) = \Omega^i(X) \otimes_\mathbf{C} M_\mathbf{C}$ the space of $M_\mathbf{C}$-valued smooth i-forms. By the de Rham Theorem (cf. [21] IV.9.1, or [31] Appendix Theorem 2) we have

$$H^i(\Gamma \backslash \mathbf{H}_3, \widetilde{M_\mathbf{C}}) \cong H^i(\Omega^\bullet(\mathbf{H}_3, M_\mathbf{C})^\Gamma).$$

Furthermore, the de Rham cohomology groups are canonically isomorphic to relative Lie algebra cohomology groups. For the definition of the latter we refer to [6] Chapter 1. The tangent space of $\mathbf{H}_3$ at the point $K_\infty \in G_\infty/K_\infty$ can be canonically identified with $\mathfrak{g}_\infty/\mathfrak{k}_\infty$. For $g \in G_\infty$ let $\mathrm{L}_g : \mathbf{H}_3 \to \mathbf{H}_3$ be the left-translation by g and $\mathrm{D}_{\mathrm{L}_g}$ the differential of this map. Assume that the $G(\mathbf{Q})$-action on $M_\mathbf{C}$ extends to a representation of G_∞. Let $\omega_{M_\mathbf{C}} : Z(\mathbf{R}) \to \mathbf{C}^*$ be the character describing the action on $M_\mathbf{C}$ and write $C^\infty(\Gamma \backslash \mathrm{GL}_2(\mathbf{C}))(\omega_{M_\mathbf{C}}^{-1})$ for those functions in $C^\infty(\Gamma \backslash \mathrm{GL}_2(\mathbf{C}))$ on which translation by elements in $Z(\mathbf{R})$ acts via $\omega_{M_\mathbf{C}}^{-1}$.

We can then identify the $\mathbf{C}$-vector spaces

$$\Omega^i(\mathbf{H}_3, M_\mathbf{C})^\Gamma \cong \mathrm{Hom}_{K_\infty}(\Lambda^i(\mathfrak{g}_\infty/\mathfrak{k}_\infty), C^\infty(\Gamma \backslash \mathrm{GL}_2(\mathbf{C}))(\omega_{M_\mathbf{C}}^{-1}) \otimes M_\mathbf{C}),$$

by mapping an $M_\mathbf{C}$-valued differential form $\tilde{\omega}$ to the $(\mathfrak{g}, K_\infty)$-cocycle ω given by $\omega(g)(\theta_1 \wedge \ldots \wedge \theta_i) := g^{-1}.\tilde{\omega}(gK_\infty)(\mathrm{D}_{\mathrm{L}_g}(\theta_1), \ldots, \mathrm{D}_{\mathrm{L}_g}(\theta_i))$. The differentials of the complexes corresponds and we get (cf. [6] VII Corollary 2.7)

$$H^i(\Gamma \backslash \mathbf{H}_3, \widetilde{M_\mathbf{C}}) \cong H^i(\mathfrak{g}_\infty, K_\infty, C^\infty(\Gamma \backslash \mathrm{GL}_2(\mathbf{C}))(\omega_{M_\mathbf{C}}^{-1}) \otimes M_\mathbf{C}).$$

Similarly, one obtains

$$H^i(S_{K_f}, \widetilde{M_\mathbf{C}}) \cong H^i(\mathfrak{g}_\infty, K_\infty, C^\infty(G(\mathbf{Q}) \backslash G(\mathbf{A})/K_f)(\omega_{M_\mathbf{C}}^{-1}) \otimes M_\mathbf{C})$$

and

$$H^i(\partial \tilde{S}_{K_f}, \widetilde{M_\mathbf{C}}) \cong H^i(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q}) \backslash G(\mathbf{A})/K_f)(\omega_{M_\mathbf{C}}^{-1}) \otimes M_\mathbf{C}).$$

For any cocycle ω we will denote by $[\omega]$ the corresponding cohomology class.

For $\Gamma\backslash\mathbf{H}_3$ and N an $\mathcal{O}[\Gamma]$-module the natural isomorphisms of the de Rham Theorem and Proposition 4 compose to give an isomorphism between de Rham cohomology and group cohomology. We state this isomorphism explicitly on the level of cocycles for degree 1 (for a proof see [2] Proposition 2.5 or, more generally, [10] Proof of Lemma 3.3.5.1):

Proposition 5. *The natural isomorphism*

$$H^1(\Omega^\bullet(\mathbf{H}_3, N_{\mathbf{C}})^\Gamma) \cong H^1(\Gamma\backslash\mathbf{H}_3, \widetilde{N_{\mathbf{C}}}) \cong H^1(\Gamma, N_{\mathbf{C}})$$

is induced by any of the following maps on closed 1-forms: For a choice of basepoint $x_0 \in \mathbf{H}_3$ *assign to a closed 1-form* $\tilde{\omega}$ *with values in* $N_{\mathbf{C}}$ *the (inhomogeneous) 1-cocycle*

$$\mathcal{G}_{x_0}(\tilde{\omega}) : \alpha \mapsto \int_{x_0}^{\alpha.x_0} \tilde{\omega}.$$

□

For each $g \in G(\mathbf{A}_f)$ with $g\widetilde{M_{\mathcal{O}}} \subset \widetilde{M_{\mathcal{O}}}$ we have the Hecke algebra action of the double coset $[K_f g K_f]$ on the cohomology groups $H^i(S_{K_f}, \widetilde{M}_R)$, $H^i(\partial\overline{S}_{K_f}, \widetilde{M}_R)$, and $H^i_!(S_{K_f}, \widetilde{M}_R)$ for any $\mathcal{O}$-algebra R (for its definition see [55] §1.4.4).

For $R = \mathbf{C}$ this can be described on the level of relative Lie algebra cohomology: If V is any $G(\mathbf{A}_f)$-module then $[K_f g K_f]$ acts on a K_f-invariant vector $v \in V$ by

$$[K_f g K_f].v = \sum_{\gamma \in K_f g K_f / K_f} \gamma.v.$$

Taking this action on $C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$, $C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$, and $C^\infty_c(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$, respectively, induces via relative Lie algebra cocycles the Hecke action on the cohomology groups. For $x \in \mathcal{O} \otimes \hat{\mathbf{Z}}$ we single out the operators

$$T_x = [K_f \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} K_f].$$

2.6. **Relative (co-)homology.** We refer to [7] II §12 and V §5 for the definitions of relative sheaf cohomology and relative Borel-Moore homology, but want to recall the following facts:

Let $\Gamma \subset G(\mathbf{Q})$ be an arithmetic subgroup , $K_f \subset G(\mathbf{A}_f)$ a compact open subgroup, and M an $F[G(\mathbf{Q})]$-module with $M_{\mathcal{O}} \subset M$ an $\mathcal{O}$-lattice as above. Then for $X = \Gamma\backslash\overline{\mathbf{H}}_3$ or $\overline{S}_{K_f}$ and $Y \subset X$ a closed subspace we have the long exact sequence (functorial in the $\mathcal{O}$-algebra R)

$$\ldots \to H^i(X, Y, \widetilde{M}_R) \to H^i(X, \widetilde{M}_R) \to H^i(Y, \widetilde{M}_R) \to H^{i+1}(X, Y, \widetilde{M}_R) \to \ldots$$

Note that for $Y = \partial\overline{S}_{K_f}$ we have $H^i(X, Y, \widetilde{M}_R) \cong H^i_c(X, \widetilde{M}_R)$.

For $Y \subset \partial\overline{S}_{K_f}$ we get the following analytic description: We say that a function $f \in C^\infty(\Gamma\backslash \mathrm{GL}_2(\mathbf{C}))$ has *moderate growth* if there exists an integer $N > 0$ and a constant $c > 0$ such that

$$|f(g)| \leq c\|g\|^N, \text{ for all } g \in \mathrm{GL}_2(\mathbf{C}).$$

For any Borel subgroup P of GL_2 defined over F we say that a function $f \in C^\infty(\Gamma\backslash \mathrm{GL}_2(\mathbf{C}))$ is *fast decreasing at* P if f has moderate growth and if there exists an integer $N > 0$ such that for every Siegel set $S \subset \mathrm{GL}_2(\mathbf{C})$ relative to P, every compact set $\omega \subset P(\mathbf{R})$, and every $r \in \mathbf{R}$ there exists a constant $c(S,\omega,r) > 0$ satisfying

$$|f(a\omega gk)| \leq c(S,\omega,r)\|a\|^N\|g\|^r,$$

for all $a \in Z, w \in \omega, g \in S \cap \mathrm{SL}_2(\mathbf{C}), k \in K_\infty$.

Let $C(\Gamma)$ be the set of Γ-conjugacy classes of Borel subgroups of GL_2 defined over F. For $\mathfrak{c} \subset C(\Gamma)$ denote by $C^\infty_{\mathfrak{c}}(\Gamma\backslash\mathrm{GL}_2(\mathbf{C}))$ the space of functions f which, together with all their derivatives $Df, D \subset U(\mathfrak{g})$, are of moderate growth and fast decreasing at every P such that $[P] \in \mathfrak{c}$. We then get the following extension of the de Rham theory recalled in Section 2.5:

Proposition 6.

$$H^1(\Gamma\backslash\overline{\mathbf{H}}_3, \bigcup_{[P]\in\mathfrak{c}} e'(P), \widetilde{M}_{\mathbf{C}}) \cong H^1(\mathfrak{g}_\infty, K_\infty, C^\infty_{\mathfrak{c}}(\Gamma\backslash\mathrm{GL}_2(\mathbf{C})) \otimes M_{\mathbf{C}}).$$

Sketch of proof. We need to show that the inclusion of the complex of differential forms with compactly supported coefficients on $\Gamma\backslash\overline{\mathbf{H}}_3 - \left(\bigcup_{[P]\in\mathfrak{c}} e'(P)\right)$ into the forms with fast decreasing coefficients induces an isomorphism in cohomology. The corresponding statement for cohomology with compact support (and for general locally symmetric spaces) is proven by Borel in [4] Theorem 5.2. Since the proof uses sheaf theory and considers stalks in the boundary it extends to our case of relative cohomology with respect to subsets of $C(\Gamma)$. □

From the comparison theorem with relative singular cohomology (see [7] X §14) we obtain the evaluation pairing

$$H^i(X,Y,\widetilde{M}_R^\vee)_{\mathrm{free}} \times H_i(X,Y,\widetilde{M}_R)_{\mathrm{free}} \to R,$$

which is perfect for any $\mathcal{O}[\frac{1}{6}]$-algebra R (see [13], Satz 3, and [18] §23). For $R = \mathbf{C}$ this pairing can be calculated by $([\omega],[\sigma]) \mapsto \int_\sigma \omega$ for ω a relative Lie algebra cocycle and σ a differentiable singular cycle. Note that for $R \subset \mathbf{C}$ a class $[\omega] \in H^i(X,Y,\widetilde{M}_{\mathbf{C}}^\vee)$ lies in $H^i(X,Y,\widetilde{M}_R^\vee)_{\mathrm{free}} \cong \mathrm{im}(H^i(X,Y,\widetilde{M}_R^\vee) \to H^i(X,Y,\widetilde{M}_{\mathbf{C}}^\vee))$ if and only if all the pairings of $[\omega]$ with homology classes in $H_i(X,Y,\widetilde{M}_R)_{\mathrm{free}}$ have values in R.

3. Eisenstein cohomology

In this section we recall Harder's construction of Eisenstein cohomology, construct explicit classes, and calculate their Hecke eigenvalues and restrictions to the boundary. We also investigate the integrality of the latter by translating to group cohomology.

3.1. **Eisenstein cocycles.** Let $m,n \in \mathbf{N}_{\geq 0}$, $k,\ell \in \mathbf{Z}$, and $M := M(m,n,k,\ell)$. We want to construct certain cohomology classes in $H^1(S_{K_f}, \tilde{M}_{\mathbf{C}})$ with nontrivial restriction to the boundary following the work of Harder in [24], [25], and [26].

Let $\phi_1,\phi_2 : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ be two Hecke characters with conductors $\mathfrak{M}_1$ and $\mathfrak{M}_2$ and of infinity type either

$$\phi_{1,\infty}(z) = z^{1-k}\overline{z}^{-n-\ell} \text{ and } \phi_{2,\infty}(z) = z^{-m-k-1}\overline{z}^{-\ell} \text{ (Case A)}$$

or

$$\phi_{1,\infty}(z) = z^{-m-k}\overline{z}^{1-\ell} \text{ and } \phi_{2,\infty}(z) = z^{-k}\overline{z}^{-n-\ell-1} \text{ (Case B).}$$

They determine a character $\phi = (\phi_1, \phi_2)$ on $T(\mathbf{Q})\backslash T(\mathbf{A})$. We put $\chi := \phi_1/\phi_2$ and denote its conductor by $\mathfrak{M}$.

Remark. These are the infinity types of Hecke characters contributing to the cohomology of the boundary as calculated in [26] §2.9 and 3.5. The two cases get swapped by the action of the Weyl group, which is defined by $w_0.(\phi_1, \phi_2) = (\phi_2|\cdot|, \phi_1|\cdot|^{-1})$, and we are in the so-called "balanced case" (cf. [26] §2.9)

For a continuous character $\eta : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ we define the induced module

$$V_{\eta,\mathbf{C}} = \left\{ \Psi : G(\mathbf{A}) \to \mathbf{C} \;\middle|\; \begin{array}{l} \Psi(bg) = \eta(b)\Psi(g),\ \forall b \in B(\mathbf{A}), \\ \Psi(gk) = \Psi(g)\ \forall k \in K_f \subset G(\mathbf{A}) \text{ compact open} \\ \Psi \text{ is } K^\infty\text{-finite on the right} \end{array} \right\}.$$

We use here the following convention: for any $\mathbf{Q}$-algebra R we consider characters η of $T(R)$ as characters of $B(R)$ by defining $\eta(b) := \eta(t)$ if $b = tu$ for $t \in T(R)$ and $u \in U(R)$. Note that the definition for V_η follows the one used in Harder's work and is not the usual unitary induction. The induced representation $V_{\eta,\mathbf{C}}$ decomposes into a product $\bigotimes_v V_{\eta_v,\mathbf{C}}$, where

$$V_{\eta_v,\mathbf{C}} = \{\Psi_v : G_0(F_v) \to \mathbf{C} | \Psi_v(b_v g_v) = \eta_v(b_v)\Psi(g_v)\, \forall b \in B_0(F_v)\}$$

denotes the local induced representations. By $V_{\eta_f,\mathbf{C}} = \otimes_{v\nmid\infty} V_{\eta_v,\mathbf{C}}$ we denote the finite part of $V_{\eta,\mathbf{C}}$. We will be interested in the operation of the Galois group $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ on $V_{\eta_f,\mathbf{C}}$ and write V_{η_f} (resp. V_{η_v} for $v \nmid \infty$) for the $\overline{\mathbf{Q}}$-subspace of $V_{\eta_f,\mathbf{C}}$ (resp. $V_{\eta_v,\mathbf{C}}$) consisting of $\overline{\mathbf{Q}}$-valued functions. Every $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ defines a σ-linear isomorphism

$$\begin{aligned} \sigma : V_{\eta_v} &\to V_{\eta_v^\sigma} \\ \Psi &\mapsto \Psi^\sigma, \end{aligned}$$

where for each $g \in G_0(F_v)$ we define $\Psi^\sigma(g) := \Psi(g)^\sigma$. By [56], ch. I.2 (see also [44] p. 94) we have $V_{\eta_v,\mathbf{C}} = V_{\eta_v} \otimes \mathbf{C}$ which implies $V_{\eta_f,\mathbf{C}} = V_{\eta_f} \otimes \mathbf{C}$. The Galois action on V_{η_f} is defined similarly to that on V_{η_v}.

Given ϕ of infinity type (A) or (B), $\Psi \in V_{\phi_f}^{K_f}$, and an appropriate open compact subgroup $K_f \subset G(\mathbf{A}_f)$ we will first define a boundary cohomology class

$$[\omega_0(\phi, \Psi)] \in H^1(\partial\overline{S}_{K_f}, \widetilde{M}_{\mathbf{C}})$$

and then an Eisenstein cohomology class

$$[\mathrm{Eis}(\phi, \Psi)] \in H^1(S_{K_f}, \widetilde{M}_{\mathbf{C}}).$$

Harder describes the cohomology of the boundary as a $G(\mathbf{A}_f)$-module in Theorem 1 of [26]:

$$H^1(\partial\overline{S}_{K_f}, \widetilde{M}_{\overline{F}}) \cong \bigoplus_{\substack{\phi : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^* \\ \text{of infinity type (A)}}} \left(V_{\phi_f}^{K_f} \oplus V_{w_0.\phi_f}^{K_f} \right). \tag{2}$$

By the proof of Theorem 2 of [26] (see also Proposition 2.12 of [2]) relative Lie algebra cocycles giving rise to non-trivial cohomology classes in

$$H^1(\partial\overline{S}_{K_f}, \widetilde{M}_{\mathbf{C}}) \cong H^1(\mathfrak{g}_\infty, K_\infty, C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_{M_{\mathbf{C}}}^{-1}) \otimes M_{\mathbf{C}})$$

can be described by certain elements in

$$\mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, V_{\phi,\mathbf{C}}^{K_f} \otimes M_{\mathbf{C}}) \cong (\check{\mathfrak{p}}_{\mathbf{C}} \otimes_{\mathbf{C}} M_{\mathbf{C}} \otimes_{\mathbf{C}} V_{\phi,\mathbf{C}}^{K_f})^{K_\infty}$$

using the map in relative Lie algebra cohomology induced by the embedding

$$V_{\phi,\mathbf{C}}^{K_f} \hookrightarrow C^\infty(B(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_{M_{\mathbf{C}}}^{-1}).$$

Recall $|\alpha| : B(\mathbf{A}) \to \mathbf{C}^*$ from Section 2.2. Following [26] p. 80 and [40] p. 101 we define

$$\omega_z(\cdot, \phi, \Psi) : G(\mathbf{A}) \to \check{\mathfrak{p}}_{\mathbf{C}} \otimes_{\mathbf{C}} M_{\mathbf{C}}$$

for $z \in \mathbf{C}$ and $\Psi \in V^{K_f}_{\phi_f|\alpha|_f^{z/2},\mathbf{C}}$ as

$$\begin{aligned} (3) \qquad & \omega_z(g,\phi,\Psi) := \omega(b_\infty k_\infty \cdot g_f, \phi|\alpha|^{z/2}, \Psi) = \\ = \quad & (\phi_\infty \cdot |\alpha|_\infty^{z/2})(b_\infty) \cdot \Psi(g_f) \begin{cases} k_\infty^{-1} . (\check{S}_+ \otimes Y^m \overline{X}^n) & \text{Case (A)}, \\ k_\infty^{-1} . ((-\check{S}_-) \otimes X^m \overline{Y}^n) & \text{Case (B)}. \end{cases} \end{aligned}$$

Here K_∞ acts on $\mathfrak{p}_{\mathbf{C}}$ by the adjoint action. By [24] Lemma 1.5.2 ω_0 is a relative Lie algebra 1-cocycle. We write $[\omega_0(\phi,\Psi)]$ both for the corresponding cohomology class in $H^1(\mathfrak{g}_\infty, \mathfrak{k}_\infty, V_\phi \otimes M_{\mathbf{C}})$ as well as its non-trivial image in $H^1(\partial \overline{S}_{K_f}, \widetilde{M_{\mathbf{C}}})$.

We now have for $\mathrm{Re}(z) \gg 0$ an operator

$$\mathrm{Eis} : V^{K_f}_{\phi_f|\alpha|_f^{z/2},\mathbf{C}} \to \mathcal{A}(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)$$

given by the formula

$$\Psi \mapsto \mathrm{Eis}(\Psi)(g) = \sum_{\gamma \in B(\mathbf{Q})\backslash G(\mathbf{Q})} \Psi(\gamma g).$$

This can be meromorphically continued to all $z \in \mathbf{C}$. Via the map on cocycles the operator induces a map in cohomology. Define $\mathrm{Eis}(\phi|\alpha|^{z/2}, \Psi) := \mathrm{Eis}(\omega_z(\phi,\Psi))$ for $\Psi \in V^{K_f}_{\phi_f|\alpha|_f^{z/2},\mathbf{C}}$.

Harder shows in [26] Theorem 2 that for $z = 0$ we get a holomorphic closed form. For $g \in G(\mathbf{A})$ and $A \in \mathfrak{g}_\infty/\mathfrak{k}_\infty$ we use the notation $\mathrm{Eis}(g,\phi,\Psi)(A)$ for the Lie algebra 1-cocycle in $\mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f)(\omega_{M_{\mathbf{C}}}^{-1}) \otimes M_{\mathbf{C}})$. The corresponding de Rham 1-form in $\Omega^1(G(\mathbf{A})/K_f K_\infty \otimes M_{\mathbf{C}})^{G(\mathbf{Q})}$ is denoted by $\mathrm{Eis}(x,\phi,\Psi)(\theta_x)$ for $x \in S_{K_f}$ and $\theta_x \in T_x S_{K_f}$. We write $[\mathrm{Eis}(\phi,\Psi)]$ for the cohomology class in $H^1(S_{K_f}, \widetilde{M_{\mathbf{C}}})$. We will later drop ϕ in the argument if it is clear from the context.

3.2. **Special vectors.** We now want to single out some special vectors

$$\Psi = \otimes_{v\nmid\infty} \Psi_v \in V_{\phi_f|\alpha|_f^{z/2}}.$$

At finite places we define the following functions:

(a) For any finite place v we define Ψ_v^{new} to be the newvector spanning $V_{\phi_v|\alpha|_v^{z/2}}^{K^1(\mathfrak{P}_v^s)}$, where $\mathfrak{P}_v^s \parallel \mathfrak{M}_1\mathfrak{M}_2$. By [8] §1 $\mathfrak{P}_v^s$ is the conductor of $V_{\phi_v|\alpha|_v^{z/2}}$ and we normalize

Ψ_v^{new} by:

$$\Psi_v^{\mathrm{new}}(g) = \begin{cases} \phi_{1,v}(a)\phi_{2,v}(d)\left|\frac{a}{d}\right|_v^{z/2} & \text{if } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} k, k \in K^1(\mathfrak{P}_v^s) \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathfrak{P}_v^r \parallel \mathfrak{M}_1$.

(b) For $v \nmid \mathfrak{M}$ we also have the spherical vector $\Psi_v^0 \in V_{\phi_v|\alpha|_v^{z/2}}^{U^1(\mathfrak{M}_{1,v})}$ defined by

$$\Psi_v^0(g) = \phi_{1,v}(a)\phi_{2,v}(d)\left|\frac{a}{d}\right|_v^{z/2}\phi_{1,v}(\det(k)) \text{ for } g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} k, k \in \mathrm{GL}_2(\mathcal{O}_v).$$

Note that $\Psi_v^0 = \Psi_v^{\mathrm{new}}$ for $v \nmid \mathfrak{M}_1\mathfrak{M}_2$.

Denote by S the finite set of places where both ϕ_i are ramified, but $\chi = \phi_1/\phi_2$ is unramified. We put

$$\Psi^{\mathrm{new}}_{\phi_f|\alpha|_f^{z/2}} := \prod_{v\nmid\infty} \Psi_v^{\mathrm{new}} \in V_{\phi_f|\alpha|_f^{z/2}}^{K_f^{\mathrm{new}}}$$

$$\Psi^0_{\phi_f|\alpha|_f^{z/2}} := \prod_{v\notin S, v\nmid\infty} \Psi_v^{\mathrm{new}} \prod_{v\in S} \Psi_v^0 \in V_{\phi_f|\alpha|_f^{z/2}}^{K_f^S},$$

where

$$K_f^{\mathrm{new}} := K^1(\mathfrak{M}_1\mathfrak{M}_2)$$

and

$$K_f^S := \prod_{v\in S} U^1(\mathfrak{M}_{1,v}) \prod_{v\notin S} K^1((\mathfrak{M}_1\mathfrak{M}_2)_v).$$

The following lemmata from [2] tell us how to translate between Ψ_v^0 and Ψ_v^{new}: Let $\eta = (\eta_1, \eta_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*$ be a continuous character, e.g. $\eta = \phi|\alpha|^{z/2}$.

Lemma 7 (([2] Lemma 4.3))**.** *Let v be a place where both η_i are unramified, and $\mu : F_v^* \to \mathbf{C}^*$ a continuous character. If $\Psi_{\eta_v}^{\mathrm{new}}$ is the newvector in V_{η_v} and $\Psi_{\eta_v\mu}^0$ is the spherical vector in $V_{\eta_v\mu}$ then $\Psi_{\eta_v}^{\mathrm{new}}(g)\mu(\det(g)) = \Psi_{\eta_v\mu}^0(g)$ for all $g \in F_v^*$.* □

Lemma 8 (([2] Lemma 4.4))**.** *Let v be a place where both η_i are unramified, and $\mu : F_v^* \to \mathbf{C}^*$ a continuous character with conductor $\mathfrak{P}_v^r$, $r > 0$. If $\Psi_{\eta_v\mu}^0$ is the spherical vector and $\Psi_{\eta_v\mu}^{\mathrm{new}}$ the newvector in $V_{\eta_v\mu}$, then we have*

$$\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)\Psi_{\eta_v\mu}^0(g\begin{pmatrix} 1 & \frac{x}{\pi_v^r} \\ 0 & 1 \end{pmatrix}) = \mu^{-1}(-1)\frac{\eta_2}{\eta_1}(\mathfrak{P}_v^r) \cdot L_v^{-1}(\frac{\eta_1}{\eta_2}, 0) \cdot \Psi_{\eta_v\mu}^{\mathrm{new}}(g).$$

Proof. Put $\Psi''(g) := \sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*} \mu^{-1}(x)\Psi_{\eta_v\mu}^0(g\begin{pmatrix} 1 & \frac{x}{\pi_v^r} \\ 0 & 1 \end{pmatrix})$. To simplify notation we will write q for π_v^r . It is easy to check that

$$\Psi''(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} g) = \eta_1(a)\mu(a)\eta_2(d)\mu(d)\Psi''(g)$$

and we refer to [2] Lemma 4.4 for the proof of right invariance under $K^1((q^2))$. This implies that $\Psi'' \in V_{\eta_v\mu}^{K^1((q^2))}$ is a multiple of the newvector, which is nonzero only on $B_0(F_v)\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix} K^1((q^2))$. We now give the calculation of this multiple.

The non-trivial character μ on $\mathcal{O}_v^*$ descends to a non-trivial character on $(\mathcal{O}_v/\mathfrak{P}_v^r)^*$, which implies that $\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*}\mu^{-1}(x)=0$. In fact, $(\mathcal{O}_v/\mathfrak{P}_v^r)^*$ can be replaced by the subgroups $(1+\mathfrak{P}_v^n)/(1+\mathfrak{P}_v^r)$ for $n=1,\dots r-1$ if $r>1$.

We have the Iwasawa decomposition

$$\begin{pmatrix}1 & 0\\ q & 1\end{pmatrix}\begin{pmatrix}1 & \frac{x}{q}\\ 0 & 1\end{pmatrix}=\begin{pmatrix}1 & \frac{x}{q}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\frac{1}{1+x} & -\frac{x^2}{q}\\ 0 & 1+x\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{q}{1+x} & 1\end{pmatrix}.$$

(This works for all x in our sum if we avoid $x=-1$ in our choice of representatives for $x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*$). We obtain

$$\Psi''(\begin{pmatrix}1 & 0\\ \pi_v^r & 1\end{pmatrix})=\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*}\mu^{-1}(x)(\eta_2/\eta_1)(x+1).$$

For $r=1$ we have

$$\Psi''(\begin{pmatrix}1 & 0\\ \pi_v^r & 1\end{pmatrix})=\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v)^*,(x+1)\neq\mathfrak{P}_v}\mu^{-1}(x)+\mu^{-1}(\pi_v-1)(\eta_2/\eta_1)(\pi_v).$$

Using that $\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v)^*}\mu^{-1}(x)=0$ this equals

$$\mu^{-1}(\pi_v-1)\left((\eta_2/\eta_1)(\pi_v)-1\right)=\mu^{-1}(-1)(\eta_2/\eta_1)(\mathfrak{P}_v)\cdot L_v^{-1}(\eta_1/\eta_2,0).$$

For $r>1$ we have

$$\Psi^0_{\eta_v\mu}(\begin{pmatrix}1 & 0\\ \pi_v^r & 1\end{pmatrix})=\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*,(x+1)\notin\mathfrak{P}_v}\mu^{-1}(x)+\sum_{x\in\mathcal{O}_v/\mathfrak{P}_v^{r-1}}\mu^{-1}(x\pi_v-1)(\eta_2/\eta_1)(x\pi_v).$$

We rewrite the second sum as

$$\mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^r)+\mu^{-1}(-1)\sum_{n=1}^{r-1}(\eta_2/\eta_1)(\pi_v^n)\sum_{u\in(\mathcal{O}_v/\mathfrak{P}_v^{r-n})^*}\mu^{-1}(1+\pi_v^n u).$$

Let $S_n:=\sum_{u\in(\mathcal{O}_v/\mathfrak{P}_v^{r-n})^*}\mu^{-1}(1+\pi_v^n u)$. Now we make the following observation: Since

$$\sum_{y\in 1+\mathfrak{P}_v^m}\mu^{-1}(y)=0$$

for $m=1,\dots,r-1$ by our initial remark we have

$$S_n=\sum_{w\in\mathcal{O}_v/\mathfrak{P}_v^{r-n}}\mu^{-1}(1+\pi_v^n w)-\sum_{w\in\mathcal{O}_v/\mathfrak{P}_v^{r-n-1}}\mu^{-1}(1+\pi_v^{n+1}w)=\begin{cases}0 & \text{if } n\le r-2,\\ -1 & \text{if } n=r-1.\end{cases}$$

We are left to evaluate

$$\begin{aligned}\Psi''(\begin{pmatrix}1 & 0\\ \pi_v^r & 1\end{pmatrix}) &= \left(\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^r)^*,(x+1)\notin\mathfrak{P}_v}\mu^{-1}(x)\right)\\ &+ \mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^r)-\mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^{r-1}).\end{aligned}$$

The sum in brackets turns out to be zero as well, since

$$0=\sum_{x\not\equiv-1\mod\mathfrak{P}_v}\mu^{-1}(x)+\mu^{-1}(-1)\sum_{n=1}^{r-1}S_n+\mu^{-1}(-1)=\sum_{x\not\equiv-1\mod\mathfrak{P}_v}\mu^{-1}(x).$$

We conclude that

$$\Psi''(\begin{pmatrix} 1 & 0 \\ \pi_v^r & 1 \end{pmatrix}) = \mu^{-1}(-1)(\eta_2/\eta_1)(\pi_v^r)\cdot(1-(\eta_1/\eta_2)(\pi_v)),$$

as desired. □

To translate from the spherical vector to the newvector we therefore also define the finite twisted sum

$$\Psi^{\mathrm{twist}}_{\phi_f|\alpha|_f^{z/2}} := \sum_{v\in S}\sum_{x\in(\mathcal{O}_v/\mathfrak{P}_v^{r_v})^*} \phi_1^{-1}(x)\Psi^0_{\phi_f|\alpha|_f^{z/2}}(g\begin{pmatrix} 1 & \frac{x}{\pi_v^{r_v}} \\ 0 & 1 \end{pmatrix}_v) \in V_{\phi_f|\alpha|_f^{z/2}}^{K_f^{\mathrm{new}}}, \tag{4}$$

where $\mathfrak{P}_v^{r_v} \parallel \mathfrak{M}_1$. Lemma 8 shows that $\Psi^{\mathrm{twist}}_{\phi_f|\alpha|_f^{z/2}}$ equals $\Psi^{\mathrm{new}}_{\phi_f|\alpha|_f^{z/2}}$ up to essentially the L-factors for $v\in S$.

3.3. **Hecke algebra action.** The definition of our special vectors was chosen such that the Eisenstein cohomology class is a Hecke eigenvector for almost all $T_{\pi_v} = [K_f^S\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}K_f^S]$ (see Section 2.5 for the definition of the Hecke operators on relative Lie algebra cocycles). By the definition of the Eisenstein cohomology class $[\mathrm{Eis}(\phi,\Psi^0_\phi)]$, it suffices to check the effect of the Hecke operator T_{π_v} on $\Psi^0_{\phi_f}\in V_{\phi_f}^{K_f^S}$. We note that for any $\Psi = \prod_w \in V_{\phi_f}^{K_f}$ with $K_f=\prod_w K_w$ the action of T_{π_v} is described by $(T_{\pi_v}.\Psi)(g) = \prod_{w\neq v}\Psi_w(g_w)\cdot(\sum_i \Psi_v(g_v\gamma_i))$, where $K_v\begin{pmatrix} \pi_v & 0 \\ 0 & 1 \end{pmatrix}K_v = \coprod_i \gamma_i K_v$. By [8] we know that $V_{\phi_v}^{K^1(\mathfrak{P}_v^s)}$ for $\mathfrak{P}_v^s \| \mathfrak{M}_1\mathfrak{M}_2$ is 1-dimensional so we get for $v\notin S$ that

$$T_{\pi_v}(\Psi^0_{\phi_f}) = a_v(\phi)\Psi^0_{\phi_f} \text{ for some } a_v(\phi)\in\mathbf{C}.$$

Lemma 9. *For $v\notin S$ the class $[\mathrm{Eis}(\phi,\Psi^0_{\phi_f})]$ is an eigenvector for T_{π_v}.*

If in addition $v\nmid\mathfrak{M}$, for example, the eigenvalue is

$$\begin{aligned} T_{\pi_v}(\Psi^0_{\phi_f})(1) &= \Psi_v^{\mathrm{new}}(\begin{pmatrix} 1 & 0 \\ 0 & \pi_v \end{pmatrix}) + \sum_{a\in\mathcal{O}_v/\mathfrak{P}_v}\Psi_v^{\mathrm{new}}(\begin{pmatrix} \pi_v & a \\ 0 & 1 \end{pmatrix}) \\ &= \phi_{2,v}(\mathfrak{P}_v) + \mathrm{Nm}(\mathfrak{P}_v)\phi_{1,v}(\mathfrak{P}_v). \end{aligned}$$

For the calculation of the eigenvalues in other cases see [2] Lemma 3.11.

3.4. **Constant terms.** We will be interested in the image of the Eisenstein cohomology classes under the restriction map

$$\mathrm{res}: H^1(S_{K_f},\widetilde{M_{\mathbf{C}}}) \cong H^1(\overline{S}_{K_f},\widetilde{M_{\mathbf{C}}}) \to H^1(\partial\overline{S}_{K_f},\widetilde{M_{\mathbf{C}}}) \cong H^1(\partial\tilde{S}_{K_f},\widetilde{M_{\mathbf{C}}}).$$

We are in the case that Harder calls "balanced" where the restriction map maps diagonally into the cohomology of the boundary, in the sense that on the level of functions

$$\mathrm{res}\circ\mathrm{Eis}: V_\phi^{K_f} \to V_\phi^{K_f}\oplus V_{w_0.\phi}^{K_f}$$

$$\Psi \mapsto \Psi + \star\frac{L(-1,\phi_1/\phi_2)}{L(0,\phi_1/\phi_2)}T_\phi\Psi \in V_\phi^{K_f}\oplus V_{w_0.\phi}^{K_f},$$

for an intertwining operator $T_\phi: V_\phi^{K_f}\to V_{w_0.\phi}^{K_f}$ and some non-zero factor $\star$.

By [24] Proposition 1.6.1, or [60] Proposition 2.2.3 (see also [50] Satz 1.10 in the case of automorphic forms) the restriction of a cohomology class in $H^1(S_{K_f}, \widetilde{M_{\mathbf{C}}})$, represented by a relative Lie algebra 1-cocycle

$$\omega \in \mathrm{Hom}_{K_\infty}(\mathfrak{g}_\infty/\mathfrak{k}_\infty, C^\infty(G(\mathbf{Q})\backslash G(\mathbf{A})/K_f) \otimes M_{\mathbf{C}}),$$

is given by the class of the constant term ω_B, where the *constant term with respect to a parabolic* P is defined as

$$\omega_P(g) = \int_{U_P(\mathbf{Q})\backslash U_P(\mathbf{A})} \omega(ug)\, du$$

for an appropriate Haar measure du. Recall the decomposition of

$$\partial \tilde{S}_{K_f} = B(\mathbf{Q})\backslash G(\mathbf{A})/K_f K_\infty$$

into its connected components given in (1). For the parabolic B^η with $\eta \in G(\mathbf{Q})$ and $\gamma \in G(\mathbf{A}_f)$ let

$$\mathrm{res}^\gamma_{B^\eta} : H^1(S_{K_f}, \widetilde{M_{\mathbf{C}}}) \to H^1(\Gamma_{\gamma,B^\eta}\backslash \mathbf{H}_3, \widetilde{M_{\mathbf{C}}})$$

be the restriction map to the boundary component $\Gamma_{\gamma,B^\eta}\backslash \mathbf{H}_3 \stackrel{j_{\eta,\gamma}}{\hookrightarrow} \partial\tilde{S}_{K_f}$. It is easy to check that

$$\mathrm{res}^\gamma_{B^\eta}[\omega] = j^*_\gamma[\omega_{B^\eta}] = j^*_{\eta,\gamma}[\omega_B].$$

It suffices therefore to calculate the constant term ω_B.

Proposition 10. (a) *If* $\Psi = \Psi^0_{\phi_f|\alpha|_f^{z/2}}$ *then*

$$\mathrm{Eis}(\Psi)_B = \omega_z(\Psi) + c(\phi, z)\omega_{-z}(\Psi^0_{w_0\cdot(\phi_f|\alpha|_f^{z/2})}),$$

where

$$c(\phi, z) = (d_F)^{-1/2}\frac{2\pi}{z+m+1}(-1)^{n+1}\cdot\frac{L(z-1,\chi)}{L(z,\chi)}\cdot\prod_{v|\mathfrak{M}} c_v(\phi, z)$$

with $c_v(\phi,z) := \int_{U_0(F_v)} \Psi^{\mathrm{new}}_v(w_0 u_v \begin{pmatrix} 1 & 0 \\ \pi_v^{t_v} & 1\end{pmatrix}) du_v$, *where* $\mathfrak{P}_v^{t_v} \parallel \mathfrak{M}_2$. *If only one of* $\{\phi_{1,v}, \phi_{2,v}\}$ *is ramified then* $c_v(\phi, z) = \frac{\phi_{2,v}(-1)}{\mathrm{Nm}(\mathfrak{M}_{1,v})}$.

(b) *If* $\Psi = \Psi^{\mathrm{new}}_{\phi_f|\alpha|_f^{z/2}}$ *then*

$$\mathrm{Eis}(\Psi)_B = \omega_z(\Psi) + c'(\phi, z)\omega_{-z}(\Psi^{\mathrm{new}}_{w_0\cdot(\phi_f|\alpha|_f^{z/2})}),$$

where

$$c'(\phi,z) = c(\phi,z)\cdot\prod_{v\in S}[(1-\mathrm{Nm}(\mathfrak{P}_v))\cdot(\chi/\mathrm{Nm})(\mathfrak{P}_v^{r_v})\cdot L_v(z,\chi)],$$

with $\mathfrak{P}_v^{r_v} \parallel \mathfrak{M}_1$.

Proof. See [26] Theorem 2(3) and [2] Proposition 3.5. ∎

3.5. **Translation to group cohomology.** The decomposition (1) and Proposition 5 imply that we have an isomorphism

$$H^1(\partial\tilde{S}_{K_f},\widetilde{M_{\mathbf{C}}})\cong\bigoplus_{[\det(\gamma)]\in\pi_0(K_f)}\ \bigoplus_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\gamma}H^1(\Gamma_{\gamma,B^\eta},M_\gamma\otimes\mathbf{C}).\tag{5}$$

The following Lemma calculates the image of the boundary cohomology class we defined in Section 3.1 under this isomorphism:

Lemma 11. *For $\phi:T(\mathbf{Q})\backslash T(\mathbf{A})\to\mathbf{C}^*$, $\Psi\in V_{\phi_f}^{K_f}$, and $\omega=\omega_0(\phi,\Psi)$ as defined in (3) the image of $[\omega]$ in $H^1(\Gamma_{\gamma,B^\eta},M_{\mathbf{C}})$ is represented by the 1-cocycle*

$$\eta_\infty^{-1}\begin{pmatrix}a&x\\0&d\end{pmatrix}\eta_\infty\mapsto\Psi(\eta_f\gamma)\begin{cases}x\int_0^1\begin{pmatrix}1&tx\\0&1\end{pmatrix}.Y^m\overline{X}^n dt & \text{if }\phi\text{ of infinity type (A)},\\ \overline{x}\int_0^1\begin{pmatrix}1&tx\\0&1\end{pmatrix}.X^m\overline{Y}^n dt & \text{if }\phi\text{ of infinity type (B)}.\end{cases}$$

(Here we denote by η_f and η_∞ the images of $\eta\in G(\mathbf{Q})$ in $G(\mathbf{A}_f)$ and G_∞, respectively.)

Proof. Put $P=B^\eta$. Recall from Proposition 5 that for the basepoint $x_0=\eta_\infty^{-1}K_\infty$ the image of $[\omega]$ is represented by the cocycle given on U^η by

$$\mathcal{G}_{x_0}(j^*_{\eta,\gamma}(\tilde{\omega}))(\eta_\infty^{-1}\begin{pmatrix}1&x\\0&1\end{pmatrix}\eta_\infty)=\int_{\eta_\infty^{-1}K_\infty}^{\eta_\infty^{-1}\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)K_\infty}j^*_{\eta,\gamma}(\tilde{\omega})=\int_{(K_\infty,\eta_f\gamma)}^{(\left(\begin{smallmatrix}1&x\\0&1\end{smallmatrix}\right)K_\infty,\eta_f\gamma)}\tilde{\omega}.$$

Here $\tilde{\omega}\in\Omega^1(G(\mathbf{A})/K_fK_\infty\otimes M_{\mathbf{C}})^{B(\mathbf{Q})}$ is the closed 1-form associated to ω, given by

$$\tilde{\omega}(x)(T):=g_\infty.\omega(g_\infty,g_f)(\mathrm{D}_{\mathrm{L}_{g_\infty^{-1}}}T)$$

for $x=(x_\infty,g_f)=(g_\infty K_\infty,g_f)\in\mathbf{H}_3\times G(\mathbf{A}_f)/K_f$ and $T\in T_{x_\infty}\mathbf{H}_3$, independent of the choice of g_∞. To calculate the path integral we apply the following lemma, adapted from [58]:

Lemma 12 (([58] Lemma 5.1))**.** *Given $h:\mathbf{R}\to G_\infty$ a differentiable homomorphism and $g_\infty\in G_\infty$, $g_f\in G(\mathbf{A}_f)/K_f$, define $c:\mathbf{R}\to\mathbf{H}_3$ by $c(t):=h(t)\cdot g_\infty\cdot K_\infty$. For $a_0,a_1\in\mathbf{R}$ let $y_i:=c(a_i)$ and denote $\dot{h}:=(Dh)_0T_0\in\mathfrak{g}_\infty$. Then one has the following equality:*

$$\int_{y_0}^{y_1}\tilde{\omega}=\int_{a_0}^{a_1}(h(t)g_\infty).\omega(h(t)g_\infty,g_f)(g_\infty^{-1}\dot{h}g_\infty)dt.$$

We take $y_0=K_\infty$, $g_\infty=1$, $g_f=\eta_f\gamma$, $h(t)=\begin{pmatrix}1&xt\\0&1\end{pmatrix}\in G_\infty$, $a_0=0$, $a_1=1$, and obtain:

$$\mathcal{G}_{x_0}(j^*_{\eta,\gamma}(\tilde{\omega}))(\eta_\infty^{-1}\begin{pmatrix}1&x\\0&1\end{pmatrix}\eta_\infty)=\int_0^1(\begin{pmatrix}1&tx\\0&1\end{pmatrix}).\omega(\begin{pmatrix}1&tx\\0&1\end{pmatrix},\eta_f\gamma)(\begin{pmatrix}0&x\\0&0\end{pmatrix})dt.$$

One calculates that for $x\in\mathbf{C}$

$$xS_+-\overline{x}S_-=\begin{pmatrix}0&x/2\\\overline{x}/2&0\end{pmatrix}\equiv\begin{pmatrix}0&x\\0&0\end{pmatrix}\mod\mathfrak{k}_\infty.$$

To complete the proof one checks that $\mathcal{G}_{x_0}(j^*_{\eta,\gamma}(\tilde{\omega}))$ is always zero on $\eta_\infty^{-1}\begin{pmatrix}a&0\\0&d\end{pmatrix}\eta_\infty$ since ω vanishes along $H\in\mathfrak{p}_{\mathbf{C}}$. □

3.6. **Rationality of Eisenstein cohomology class and integrality of constant term.** Harder proves that the transcendentally defined Eisenstein operator is, in fact, rational, i.e., that Eis is defined over $\overline{\mathbf{Q}}$ so that we have

$$V_{\phi_f}^{K_f} \to H^1(S_{K_f}, \widetilde{M}_{\overline{\mathbf{Q}}})$$
$$\Psi \mapsto [\mathrm{Eis}(\phi, \Psi)]$$

and for $\sigma \in \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ and $\Psi \in V_{\phi_f}^{K_f}$ the equation $[\mathrm{Eis}(\phi,\Psi)]^\sigma = [\mathrm{Eis}(\phi^\sigma, \Psi^\sigma)]$ holds (see [26] Corollary 4.2.1(a); the proof uses the "Multiplicity one Theorem" for automorphic forms and the vanishing of residual interior cohomology in our case).

We are interested in the p-adic properties of the Eisenstein cohomology class. From now on, let $\phi = (\phi_1, \phi_2) : T(\mathbf{Q})\backslash T(\mathbf{A}) \to C^*$ denote a continuous character of infinity type (A) for which the conductors of both ϕ_i are coprime to (p), and let $m \geq n$ (recall that $\chi := \phi_1/\phi_2$). Let $\mathcal{O}_\phi$ denote the ring of integers in the finite extension F_ϕ of $F_\mathfrak{p}$ obtained by adjoining the values of the finite part of both ϕ_i and $L^{\mathrm{alg}}(0, \chi)$. For the definition of the latter and its p-adic properties see Theorem 3. Then the above discussion shows:

Proposition 13. *For* $\Psi = \Psi^0_{\phi_f}$ *or* $\Psi^{\mathrm{new}}_{\phi_f}$ *we have*

$$[\mathrm{Eis}(\phi, \Psi)] \in H^1(S_{K_f}, \widetilde{M}_{F_\phi})$$

for $K_f = K_f^S$ *or* $K_f = K_f^{\mathrm{new}}$*, respectively.*

Definition 14. If $\mathcal{O}_L$ is the ring of integers in a local field L over $F_\mathfrak{p}$, define for any $c \in H^1(S_{K_f}, \widetilde{M}_L)$ the *denominator (ideal)*

$$\delta(c) = \{a \ \in \mathcal{O}_L : ac \in H^1(S_{K_f}, \widetilde{M}_{\mathcal{O}_L})_{\mathrm{free}}\}.$$

Here we identify $H^1(S_{K_f}, \widetilde{M}_{\mathcal{O}_L})_{\mathrm{free}}$ with $\mathrm{im}(H^1(S_{K_f}, \widetilde{M}_{\mathcal{O}_L}) \to H^1(S_{K_f}, \widetilde{M}_L))$.

In this paper we prove (under certain conditions on the conductors of the characters ϕ_i) that $L^{\mathrm{alg}}(0,\chi)\cdot\mathcal{O}_\phi$ is a lower bound on the denominator of the Eisenstein cohomology class.

For the arithmetic of the Eisenstein cohomology class its constant term is, of course, of great importance. Put $N := M(m, n, -m-k, -n-\ell)$. Note that since $N^\vee \cong M = M(m,n,k,\ell)$ as $G(\mathbf{Q})$-modules, $[\mathrm{Eis}(\phi, \Psi^0_{\phi_f})]$ can also be interpreted as a cohomology class for the local system $\widetilde{N}^\vee_{\mathbf{C}}$. The following result shows that under certain conditions its constant term is integral already with respect to the lattice $(N_\mathcal{O})^\vee \subset M_\mathcal{O} \subset M$. We first prove the following Lemma:

Lemma 15. *Assume in addition that* $p > m+1$*. We have*

$$[\omega_0(\phi, \Psi^0_\phi)], [\omega_0(w_0.\phi, \Psi^0_{w_0.\phi})] \in H^1(\partial \tilde{S}_{K_f^S}, (\widetilde{N_{\mathcal{O}_\phi}})^\vee)_{\mathrm{free}}.$$

Proof. We recall from Section 2.5 that we have an R-functorial isomorphism

$$H^1(\partial\tilde{S}_{K_f^S}, \widetilde{N}^\vee_R) \cong \bigoplus_{[\gamma]\in\pi_0(K_f^S)} \bigoplus_{[\eta]\in\mathbf{P}^1(F)/\Gamma_\gamma} H^1(\Gamma_{\gamma,B^\eta}, (N_\gamma)^\vee \otimes R).$$

We will show that for $[\omega_0(\phi, \Psi^0_\phi)] \in H^1(\partial\tilde{S}_{K_f^S}, \widetilde{N}^\vee_{\mathbf{C}})$ the restrictions to each of the summands on the right hand side lie in the image of $H^1(\Gamma_{\gamma,B^\eta}, (N_\gamma)^\vee \otimes \mathcal{O}_\phi)$

inside $H^1(\Gamma_{\gamma,B^\eta},(N_\gamma)^\vee\otimes\mathbf{C})$. We showed in Lemma 11 that for each γ and η this restriction is given by

$$\eta^{-1}\begin{pmatrix}a & x\\ 0 & b\end{pmatrix}\eta\mapsto x\Psi^0_{\phi_f}(\eta_f\gamma)\int_0^1\begin{pmatrix}1 & tx\\ 0 & 1\end{pmatrix}.Y^m\overline{X}^n dt.$$

To check the integrality we choose representatives γ and η whose $\mathfrak{p}$- and $\overline{\mathfrak{p}}$-components are units (i.e., they are elements of $\mathrm{GL}_2(\mathcal{O}_p):=\prod_{v|p}\mathrm{GL}_2(\mathcal{O}_v)$). This is possible for γ by the Chebotarev density theorem. For η this follows from

$$\mathrm{GL}_2(F_p):=\prod_{v|p}\mathrm{GL}_2(F_v)=B_0(F)\mathrm{GL}_2(\mathcal{O}_p).$$

For such γ we get $N_\gamma^\vee\otimes\mathcal{O}_\phi=(N_{\mathcal{O}_\phi})^\vee$ and we need to show that the values of the group cocycle satisfy that the coefficient of $X^iY^{m-i}\overline{X}^j\overline{Y}^{n-j}$ lies in $\binom{m}{i}\binom{n}{j}\mathcal{O}_\phi$. Firstly, $\Psi^0_{\phi_f}(\eta_f\gamma)\in\mathcal{O}_\phi^*$ by the definition of the spherical vector at places away from the conductors of the ϕ_i together with Lemma 2. We also note that $\Gamma_\gamma\cap G(\mathbf{Q}_p)\subset\mathrm{GL}_2(\mathcal{O}_p)$, so x lies in $\mathcal{O}_\phi$. Lastly, the integral provides us with the correct coefficients for the monomials up to p-adic units if we assume $p>m+1$.

A similar argument for $[\omega_0(w_0.\phi,\Psi^0_{w_0.\phi})]$ proves integrality if $p>n+1$ (and $m\geq n$ by assumption). □

We proved in Proposition 10 that $\mathrm{Eis}(\Psi^0_{\phi_f})_B=\omega_0(\Psi^0_{\phi_f})+c(\phi,0)\omega_0(\Psi^0_{w_0.\phi_f})$. By [26] Corollary 4.2.2 we know that $c(\phi,0)\in F_\phi$. The following proposition analyzes conditions when $c(\phi,0)$ (and so by the preceding Lemma the constant term of the Eisenstein class) is integral.

Proposition 16. *Assume in addition that $p>m+1$ and that the conductors of ϕ_1 and $\chi=\phi_1/\phi_2$ are coprime.*

(a) *The constant term $[\mathrm{Eis}(\phi,\Psi^0_{\phi_f})_B]$ is integral if and only if $\frac{L^{\mathrm{alg}}(-1,\chi)}{L^{\mathrm{alg}}(0,\chi)}\in\mathcal{O}_\phi$.*

(b) *For $m=n$, we have $[\mathrm{Eis}(\phi,\Psi^0_{\phi_f})_B]\in H^1(\partial\overline{S}_{K_f^S},\widetilde{(N_{\mathcal{O}_\phi})^\vee})_{\mathrm{free}}$ if and only if $\frac{L(0,\overline{\chi})}{L(0,\chi)}\in\mathcal{O}_\phi$.*

(c) *If $m=n$ and $\chi^c(x):=\chi(\overline{x})=\overline{\chi}(x)$ for all $x\in\mathbf{A}_F^*$ then*

$$[\mathrm{Eis}(\phi,\Psi^0_{\phi_f})_B]\in H^1(\partial\overline{S}_{K_f^S},\widetilde{(N_{\mathcal{O}_\phi})^\vee})_{\mathrm{free}}.$$

Remark.

1. In [2] we called characters χ satisfying $\chi^c=\overline{\chi}$ *anticyclotomic.* These include, in particular, the everywhere unramified characters.
2. As explained in the introduction, the integrality of the constant term of the Eisenstein cohomology class (together with the bound on the denominator) is interesting for finding congruences between cuspidal and Eisenstein cohomology classes controlled by the L-value $L^{\mathrm{alg}}(0,\chi)$. Since cuspidal cohomology classes only exist for $m=n$ by Wigner's Lemma (see, e.g. [28] §3) these are the coefficient systems we are most interested in.
3. By considering $c(\phi,0)\overline{c(\phi,0)}$ and using $\overline{c(\phi,0)}=c(\overline{\phi},0)$ (cf. [26] Corollary 4.2.2) one can show that the quotient of L-values in (b) is always integral for either $\mathfrak{p}$ or $\overline{\mathfrak{p}}$.

Proof. One easily checks that

$$c(\phi,0)=\frac{L^{\mathrm{alg}}(0,\chi|\cdot|^{-1})}{L^{\mathrm{alg}}(0,\chi)}(-1)^{n+1}\prod_{v|\mathfrak{M}}c_v(\phi,0).$$

Since $\mathfrak{M}$ is coprime to p and the conductor of ϕ_1 we have $c_v(\phi,0)=\frac{\phi_{2,v}(-1)}{\mathrm{Nm}(\mathfrak{M}_{1,v})}\in\mathcal{O}_\phi^*$.

For (b) we can apply the functional equation (cf. [11] p. 37) and use that $m=n$, and therefore $\chi\overline{\chi}=|\cdot|^2$, to obtain

$$c(\phi,0)=\frac{L(0,\overline{\chi})}{L(0,\chi)}(-1)^{n+1}W(\chi)\sqrt{\mathrm{Nm}(\mathfrak{M})}\prod_{v|\mathfrak{M}}c_v(\phi,0),$$

where $W(\chi)$ is the Artin root number for χ (see Section 2.3). By the assumption on the conductor of χ both $\sqrt{\mathrm{Nm}(\mathfrak{M})}$ and the root number $W(\chi)$ lie in $\mathcal{O}_\phi^*$.

Lastly, if $\chi^c=\overline{\chi}$ then we have $L(0,\overline{\chi})=L(0,\chi^c)=L(0,\chi)$, so $\frac{L(0,\overline{\chi})}{L(0,\chi)}=1$. □

4. Toroidal integral

In this section we calculate the integral of twists of the Eisenstein cocycle defined in Section 3 against certain relative cycles.

4.1. **Definition of relative cycles.** Recall that strong approximation implies that S_{K_f} is the finite disjoint union of connected components indexed by a set of representative $\{[\xi]\}$ for $\pi_0(K_f)$ with $\xi\in\mathbf{A}_{F,f}^*$. The connected component lying above ξ is given by $\Gamma_\xi\backslash\mathbf{H}_3$ with $\Gamma_\xi:=G(\mathbf{Q})\cap\begin{pmatrix}1&0\\0&\xi\end{pmatrix}K_f\begin{pmatrix}1&0\\0&\xi\end{pmatrix}^{-1}$ and we defined in Section 2.2 an embedding $j_\xi:=j_{\begin{pmatrix}1&0\\0&\xi\end{pmatrix}}:\Gamma_\xi\backslash\mathbf{H}_3\hookrightarrow S_{K_f}$. For each $\xi\in\mathbf{A}_{F,f}^*$ let

$$\sigma_\xi=j_\xi\circ\tau:\mathbf{C}^*\to S_{K_f},$$

where $\tau:\mathbf{C}^*\to G_\infty:z\mapsto\begin{pmatrix}1&0\\0&z\end{pmatrix}$. We will use σ_ξ to denote both the map to $G(\mathbf{A})$ and the induced map to S_{K_f}. We consider the path in S_{K_f} given by $\sigma_\xi|_{\mathbf{R}_{>0}^*}$, which is the restriction of a path (also denoted by σ_ξ) in $\overline{S}_{K_f}$: for the component $\Gamma_\xi\backslash\overline{\mathbf{H}}_3$ that path is $\sigma_\xi:[0,\infty]\to\Gamma_\xi\backslash\overline{\mathbf{H}}_3\subset\overline{S}_{K_f}:t\mapsto j_\xi((t,0))$. The paths σ_ξ are not 1-cycles in $\overline{S}_{K_f}$. They are, however, relative cycles in $C_1(\Gamma_\xi\backslash\overline{\mathbf{H}}_3,\partial(\Gamma_\xi\backslash\overline{\mathbf{H}}_3),\mathbf{Z})$ (cf. [40] §5.2). Since the endpoints lie in the ∞- and 0-cusps (use $\begin{pmatrix}1&0\\0&\frac{1}{s}\end{pmatrix}K_\infty=w_0\begin{pmatrix}1&0\\0&s\end{pmatrix}K_\infty$) they are, in fact, relative cycles for

$$H_1(\Gamma_\xi\backslash\overline{\mathbf{H}}_3,\Gamma_{\xi,B}\backslash e(B)\cup\Gamma_{\xi,B^w}\backslash e(B^w),\mathbf{Z}).$$

Put

$$Q_{m',n'}:=X^{m-m'}Y^{m'}\overline{X}^{n-n'}\overline{Y}^{n'}\in N_{\mathcal{O}}\text{ for }0\le m'\le m,\ 0\le n'\le n.$$

Let

$$\theta_{m',n'}:F^*\backslash\mathbf{A}_F^*\to\mathbf{C}^*\tag{6}$$

be a Hecke character with $\theta_{m',n',\infty}(z) = z^{m-m'+k}\overline{z}^{n-n'+\ell}$ and conductor $\mathfrak{N}$ coprime to (pd_F) and the conductors of ϕ_1 and ϕ_2 such that $\#(\mathcal{O}/\mathfrak{N})^*$ is also coprime to p. Denote by T the set of places where $\theta_{m',n'}$ is ramified.

For any $\xi \in \mathbf{A}_{F,f}^*$ consider now the chain

$$(\sigma_\xi \otimes Q_{m',n'}) \cdot \theta_{m',n'}(\xi) \in C_1(\Gamma_\xi \backslash \mathbf{H}_3, N_\xi \otimes \mathcal{O}_\theta),$$

with $N_\xi := N_{\begin{pmatrix}1 & 0\\ 0 & \xi\end{pmatrix}} \cong j_\xi^* \widetilde{N}_{\mathcal{O}}$ and $\mathcal{O}_\theta$ the ring of integers in the finite extension F_θ of F_ϕ obtained by adjoining the values of $\theta_{m',n',f}$. Here we use Lemma 2 to check the integrality of the chain. Since chains for S_{K_f} are defined as $G(\mathbf{Q})$-coinvariants of chains in $G(\mathbf{A})/K_f K_\infty$ the sum

$$\sum_{[\xi]\in\pi_0(K_f)} (\sigma_\xi \otimes Q_{m',n'}) \cdot \theta_{m',n'}(\xi) \in C_1(S_{K_f}, N_{\mathcal{O}} \otimes \mathcal{O}_\theta)$$

is independent of the choice of representatives ξ for the connected components. As observed above this chain is, in fact, a relative cycle with endpoints in the ∞ and 0-cusps of all the connected components $\Gamma_\xi\backslash\overline{\mathbf{H}}_3$.

4.2. **Twisted version of Eisenstein cocycle.** We now define the following twisted version of the Eisenstein cocycle: Let $\eta = (\eta_1, \eta_2) : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ be a continuous character. For $\Psi \in V_{\eta_f}$ let

$$\mathrm{Eis}^\theta(g,\Psi) := \sum_{v\in T}\sum_{x\in(\mathcal{O}_v/\mathfrak{N}_v)^*} \theta_{m',n'}^{-1}(x)\mathrm{Eis}\left(g\begin{pmatrix}1 & -\frac{x}{\pi_v^{\mathrm{ord}_v\mathfrak{N}}}\\ 0 & 1\end{pmatrix}_v, \Psi\right).$$

Note that if $\Psi = \prod_v \Psi_v$ then the twisting can also be done on the level of the function, i.e.,

$$\mathrm{Eis}^\theta(g,\Psi) = \mathrm{Eis}(g,\Psi^\theta),$$

where $\Psi^\theta(g) := \prod_{v\in T}\Psi_v^\theta(g_v)\prod_{v\notin T}\Psi_v(g_v)$ and

$$\Psi_v^\theta(g_v) := \sum_{v\in T}\sum_{x\in(\mathcal{O}_v/\mathfrak{N}_v)^*} \theta_{m',n'}^{-1}(x)\Psi_v\left(g\begin{pmatrix}1 & -\frac{x}{\pi_v^{\mathrm{ord}_v\mathfrak{N}}}\\ 0 & 1\end{pmatrix}_v\right).$$

Lemmata 7 and 8 imply:

Lemma 17. *For $v \in T$ we have*

$$\Psi_{\eta,v}^{\mathrm{new},\theta}(g) = \Psi_{\eta\theta,v}^{\mathrm{new}}(g)\theta_v^{-1}(-\det(g)) \cdot (\eta_2/\eta_1)(\pi_v^{\mathrm{ord}_v\mathfrak{N}}) \cdot L_v^{-1}(\eta_1/\eta_2, 0).$$

Now consider $\eta = \phi|\alpha|^{z/2}$. We note that if $[\mathrm{Eis}(g,\Psi)] \in H^1(S_{K_f}, \widetilde{M}_{F_\phi})$ then

$$[\mathrm{Eis}^\theta(g,\Psi)] \in H^1(S_{(K_f)^\theta}, \widetilde{M}_{F_\theta}),$$

where for $K_f = \prod_v K_v$ we define $(K_f)^\theta = \prod_v K_v'$ with $K_v' = K_v$ for $v \notin T$ and $K_v' = K_v \cap U^1(\mathfrak{N}_v)$ for $v \in T$. We will see in Lemma 20 that this twisting makes $\mathrm{Eis}^\theta(\Psi)$ a relative cocycle with respect to the 0- and ∞-cusps of each connected component (in the sense of Proposition 6).

4.3. **Calculation of the toroidal integral.** We now want to integrate $\mathrm{Eis}^\theta(\Psi)$ for $\Psi \in V_{\phi_f|\alpha|_f^{z/2}}^{K_f^{\rm new}}$ over the relative cycle defined in Section 4.1: Put $K_f^\theta := (K_f^{\rm new})^\theta$ and let

$$\begin{aligned} I(\phi,\theta,\Psi,z) &:= \sum_{[\xi]\in\pi_0(K_f^\theta)} \theta_{m',n'}(\xi) \int_{\sigma_\xi \otimes Q_{m',n'}} \mathrm{Eis}^\theta(\Psi) \\ &= \sum_{[\xi]\in\pi_0(K_f^\theta)} \theta_{m',n'}(\xi) \int_0^\infty \left\langle Q_{m',n'}, \mathrm{Eis}^\theta(\sigma_\xi(t),\Psi)(d\sigma_\xi(t\frac{\partial}{\partial t}))\right\rangle \frac{dt}{t}. \end{aligned}$$

We will first evaluate this "toroidal integral" for $\Psi = \Psi^{\rm new} := \Psi^{\rm new}_{\phi_f|\alpha|_f^{z/2}}$. Rewriting the Eisenstein cocycle as a relative Lie algebra cocycle we have

$$I(\phi,\theta,\Psi^{\rm new},z) =$$
$$= \textstyle\sum_{[\xi]\in\pi_0(K_f^\theta)} \theta_{m',n'}(\xi) \int_0^\infty \left\langle \begin{pmatrix} 1 & 0 \\ 0 & t^{-1}\end{pmatrix}.Q_{m',n'}, \mathrm{Eis}^\theta(\sigma_\xi(t),\Psi^{\rm new})(d\sigma_\xi(\frac{\partial}{\partial t}|_{t=1}))\right\rangle \frac{dt}{t}.$$

Using the K_∞-invariance of the Eisenstein cocycle, the argument on pp.107/8 in [40] shows that this equals

$$\sum_{[\xi]\in\pi_0(K_f^\theta)} \theta_{m',n'}(\xi) \int_0^{2\pi}\int_0^\infty$$
$$\left\langle \begin{pmatrix} 1 & 0 \\ 0 & u^{-1}\end{pmatrix}.Q_{m',n'}, \mathrm{Eis}^\theta(\sigma_\xi(u),\Psi^{\rm new})(\mathrm{Ad}(\sigma_\xi(e^{-i\varphi}))d\sigma_\xi(\tfrac{\partial}{\partial t}|_{t=1}))\right\rangle \tfrac{dt}{t}\wedge\tfrac{d\varphi}{2\pi}$$

with $u = te^{i\varphi} \in \mathbf{C}^*$. Since $d\sigma_\xi(\frac{\partial}{\partial t}|_{t=1}) = \frac{H}{2}$, this equals

$$\sum_{[\xi]\in\pi_0(K_f^\theta)} \theta_{m',n'}(\xi) \int_{\mathbf{C}^*} \theta_{m',n',\infty}(x_\infty) \left\langle Q_{m',n'}, \mathrm{Eis}^\theta(\begin{pmatrix} 1 & 0 \\ 0 & \xi x_\infty\end{pmatrix}, \Psi^{\rm new})(\frac{H}{2})\right\rangle d^*x_\infty$$

with $d^*x_\infty := \frac{i}{4\pi}\frac{dx_\infty\wedge d\overline{x}_\infty}{x_\infty\overline{x}_\infty}$.

Note that $\det(K_f^\theta) = \prod_{v\notin T}\mathcal{O}_v^*\prod_{v\in T}(1+\mathfrak{N}_v)$. Normalize a Haar measure $d^*x = d^*x_\infty\prod_{v\nmid\infty} d^*x_v$ on $\mathbf{A}_F^*$ such that for finite places $\int_{(\mathcal{O}_v/\mathfrak{N}_v)^*} d^*x_v = 1$. Using the right K_f^θ-invariance we can then write

$$I(\phi,\theta,\Psi^{\rm new},z) = \int_{F^*\backslash\mathbf{A}_F^*} \theta_{m',n'}(x) \left\langle Q_{m',n'}, \mathrm{Eis}^\theta(\begin{pmatrix} 1 & 0 \\ 0 & x\end{pmatrix}, \Psi^{\rm new})(\frac{H}{2})\right\rangle d^*x.$$

Proposition 18. *For* $\mathrm{Re}(z) \gg 0$ *the integral* $I(\phi,\theta,\Psi^{\rm new},z)$ *converges and the value is*

$$\frac{L(\frac{z}{2},\phi_1\theta_{m',n'})L(\frac{z}{2},\phi_2^{-1}\theta_{m',n'}^{-1})}{L^S(z,\phi_1/\phi_2)} \cdot \frac{\Gamma(\frac{z}{2}+m-m'+1)\Gamma(\frac{z}{2}+m'+1)}{\Gamma(z+m+2)} \cdot \#(\mathcal{O}/\mathfrak{N})^* \cdot$$
$$\cdot \frac{(-1)^{n-n'+k+\ell}}{2}\left((\theta_{m',n'}\phi_2)^{-1}(\mathfrak{M}_1\mathfrak{N})\mathrm{Nm}(\mathfrak{M}_1\mathfrak{N})^{-\frac{z}{2}}\right)\cdot(\phi_2/\phi_1)(\mathfrak{N})\mathrm{Nm}(\mathfrak{N})^z.$$

Remark. Here the factor $\phi_2^{-1}(\mathfrak{M}_{1,v})$ at places $v \in S$ stands for $\phi_{2,v}^{-r}(\pi_v)$ for the choice of uniformizer π_v in the definition of the newvector and $\theta_{m',n'}(\mathfrak{N})$ for the product $\prod_{v\in T}\theta_{m',n'}(\pi_v^{\mathrm{ord}_v\mathfrak{N}})$ for the uniformizers π_v chosen in the definition of $\mathrm{Eis}^\theta(\Psi)$.

Proof. We start by unfolding the Eisenstein series

$$\mathrm{Eis}^\theta(g,\Psi^{\mathrm{new}}) = \sum_{\gamma\in B(\mathbf{Q})\backslash G(\mathbf{Q})} \Psi^{\mathrm{new},\theta}(\gamma g)$$

for $\mathrm{Re}(z)\gg 0$ and use analytic continuation to deduce the result for all z for which the integral converges.

Following [40] §4.5 we use a refinement of the Bruhat decomposition choosing representatives for $B(\mathbf{Q})\backslash G(\mathbf{Q})$ according to the orbits of the $T(\mathbf{Q})$-action:

$$G(\mathbf{Q}) = B(\mathbf{Q})\begin{pmatrix}1&0\\0&1\end{pmatrix}\cup B(\mathbf{Q})w_0\cup B(\mathbf{Q})\begin{pmatrix}1&0\\1&1\end{pmatrix}T_1(\mathbf{Q}),$$

where $T_1(\mathbf{Q}) = \left\{\begin{pmatrix}1&0\\0&b\end{pmatrix} : b\in F^*\right\}$. If we decompose the integral according to this sum, the integral over the first two summands vanishes, since $\omega_z(g_f b_\infty,\phi,\Psi^{\mathrm{new},\theta}) = \Psi_f^{\mathrm{new},\theta}(g_f)\omega_\infty(b_\infty)$ is zero along H (here we factor (3) as $\omega_z(g,\phi,\Psi) = \omega_\infty(g_\infty)\cdot\Psi(g_f)$). We would like to write the remaining term as

$$\int_{\mathbf{A}_F^*}\theta_{m',n'}(x)\Psi^{\mathrm{new},\theta}(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&x_f\end{pmatrix})\cdot\left\langle Q_{m',n'},\omega_\infty(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&x_\infty\end{pmatrix})(\frac{H}{2})\right\rangle d^*x.$$

This step is justified if the latter integral converges absolutely. Since the integrand decomposes by definition as a product of local functions, the integral can be written as a product of local integrals:

$$\prod_{v\nmid\infty}\int_{F_v^*}\theta_{m',n'}(x_v)\Psi_v^{\mathrm{new},\theta}(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&x_v\end{pmatrix})d^*x_v$$
$$\times\int_{F_\infty^*}\left\langle Q_{m',n'},\omega_\infty(\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&x_\infty\end{pmatrix})(\frac{H}{2})\right\rangle d^*x_\infty.$$

For each finite place v we define integers $r=r_v$, $s=s_v$ by $\mathfrak{P}_v^r\|\mathfrak{M}_1$ and $\mathfrak{P}_v^s\|\mathfrak{M}_1\mathfrak{M}_2$. We will treat the local integrals according to the following cases:

(1) v finite place, $v\notin T$, both ϕ_i unramified, i.e., $r=s=0$
(2) v finite place, ϕ_1 ramified, ϕ_2 unramfied, i.e., $r=s>0$
(3) v finite place, ϕ_1 unramified, ϕ_2 ramified, i.e., $r=0$, $s>0$
(4) v finite place, $r>0$ and $s-r>0$
(5) $v\in T$
(6) v archimedean

Before we start, we work out the Iwasawa decomposition of our argument at the finite places:

$$\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}1&0\\0&x_v\end{pmatrix} = \begin{pmatrix}1&0\\1&x_v\end{pmatrix} = \begin{cases}\begin{pmatrix}x_v&1\\0&1\end{pmatrix}\begin{pmatrix}0&-1\\1&x_v\end{pmatrix} & \text{if } \mathrm{ord}_v(x_v)\geq 0,\\ \begin{pmatrix}1&0\\0&x_v\end{pmatrix}\begin{pmatrix}1&0\\x_v^{-1}&1\end{pmatrix} & \text{if } \mathrm{ord}_v(x_v)<0.\end{cases}$$

We decompose F_v^* into a disjoint union of $\pi_v^t\mathcal{O}_{F_v}^*$ for $t\in\mathbf{Z}$.

In case (1), the integrand over $\pi_v^t\mathcal{O}_v^*$ is

$$\begin{cases}(\phi_1\theta_{m',n'})_v^t(\pi_v)|\pi_v|_v^{tz/2} & \text{if } t\geq 0,\\ (\phi_2\theta_{m',n'})_v^t(\pi_v)|\pi_v|_v^{-tz/2} & \text{if } t<0.\end{cases}$$

The integral therefore is given by two infinite sums, and since the infinity type of $\phi_1\theta_{m',n'}$ is $z^{1+m-m'}\overline{z}^{-n'}$, and that of $\phi_2^{-1}\theta_{m',n'}^{-1}$ is $z^{1+m'}\overline{z}^{n'-n}$ it converges for

$$\mathrm{Re}(z)>n'-(m-m'+1) \text{ and } \mathrm{Re}(z)>(n-n')-(m'+1)$$

and the value is

$$\begin{aligned}&\frac{1}{1-(\phi_{1,v}\theta_{m',n',v})(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2}}+\frac{(\phi_{2,v}\theta_{m',n',v})^{-1}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2}}{1-(\phi_{2,v}\theta_{m',n',v})^{-1}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-z/2}}\\ =\ &\frac{L_v(\frac{z}{2},\phi_1\theta_{m',n'})L_v(\frac{z}{2},(\phi_2\theta_{m',n'})^{-1})}{L_v(z,\phi_1/\phi_2)}.\end{aligned}$$

In case (2), the definition of the newvector Ψ_v^{new} shows that the integrand is non-zero only over $\pi_v^t\mathcal{O}_v^*$ with $t\leq -r$. The integral therefore is given by

$$\begin{aligned}&\sum_{t\geq r}(\phi_{2,v}\theta_{m',n',v})^{-t}(\pi_v)|\pi_v|_v^{tz/2}=\\ &=(\phi_{2,v}\theta_{m',n',v})^{-r}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-rz/2}\cdot L_v(\tfrac{z}{2},(\phi_2\theta_{m',n'})^{-1}).\end{aligned}$$

For case (3) we only get a non-zero contribution for $\mathrm{ord}_v(x_v)\geq 0$. Since for such x_v, $\begin{pmatrix}1&0\\1&x_v\end{pmatrix}=\begin{pmatrix}x_v&*\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}k$ with $k\in K^1(\mathfrak{P}_v^s)$, the integral equals

$$\sum_{t\geq 0}(\phi_{1,v}\theta_{m',n',v})^t(\pi_v)|\pi_v|_v^{tz/2}=L_v(\frac{z}{2},\phi_1\theta_{m',n'}).$$

In case (4), Ψ_v is non-zero only on $\left[\begin{pmatrix}1&0\\\pi_v^r&1\end{pmatrix}\right]\in B(F_v)\backslash\mathrm{GL}_2(F_v)/K^1(\mathfrak{P}_v^s)$. This means we have to have $\mathrm{ord}_v(x_v)=-r$ exactly. If $x_v=\epsilon\pi_v^{-r}$ with $\epsilon\in\mathcal{O}_v^*$ we have

$$\begin{pmatrix}1&0\\0&x_v\end{pmatrix}\begin{pmatrix}1&0\\x_v^{-1}&1\end{pmatrix}=\begin{pmatrix}1&0\\0&\epsilon\pi_v^{-r}\end{pmatrix}\begin{pmatrix}1&0\\\epsilon^{-1}\pi_v^r&1\end{pmatrix}=$$

$$\begin{pmatrix}1&0\\0&\epsilon\pi_v^{-r}\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&\epsilon^{-1}\end{pmatrix}\begin{pmatrix}1&0\\\pi_v^r&1\end{pmatrix}\begin{pmatrix}1&0\\0&\epsilon\end{pmatrix}=\begin{pmatrix}1&0\\0&\pi_v^{-r}\end{pmatrix}\begin{pmatrix}1&0\\\pi_v^r&1\end{pmatrix}\begin{pmatrix}1&0\\0&\epsilon\end{pmatrix}.$$

The integral therefore is given by

$$\int_{\mathcal{O}_v^*}(\phi_{2,v}\theta_{m',n',v})^{-r}(\pi_v)|\pi_v|_v^{rz/2}d^*\epsilon=(\phi_{2,v}\theta_{m',n',v})^{-r}(\pi_v)\mathrm{Nm}(\mathfrak{P}_v)^{-rz/2}.$$

Case (5): For $v\in T$ Lemma 17 implies that the local factor is given by

$$\theta_{m',n',v}^{-1}(-1)(\phi_2/\phi_1)(\mathfrak{N}_v)\mathrm{Nm}(\mathfrak{N}_v)^zL_v^{-1}(z,\phi_1/\phi_2)\int_{F_v^*}\Psi^{\mathrm{new}}_{(\phi_1,\phi_2)|\alpha|^{z/2}\theta}(\begin{pmatrix}1&0\\1&x_v\end{pmatrix})d^*x_v.$$

Proceeding as in case (4) and taking the normalization of the local measures into account we obtain

$$\theta_{m',n',v}^{-1}(-1)L_v^{-1}(z,\phi_1/\phi_2)\cdot(\phi_1\theta_{m',n'})_v^{-1}(\mathfrak{N})\mathrm{Nm}(\mathfrak{N}_v)^{z/2}\cdot\#(\mathcal{O}_v/\mathfrak{N}_v)^*.$$

In Case (6) the archimedean factor is

$$\frac{i}{8\pi}\int_{\mathbf{C}^*} u^{m-m'+k}\overline{u}^{n-n'+\ell}\left\langle Q_{m',n'},\omega_\infty(\begin{pmatrix}1&0\\1&u\end{pmatrix})(H)\right\rangle \frac{du\wedge d\overline{u}}{u\overline{u}}.$$

Here we denote $(\phi_\infty|\alpha|_\infty^{z/2})(b_\infty)k_\infty^{-1}.\check{S}_+$ by $\omega_\infty(b_\infty k_\infty)$, so

$$\omega_\infty(b_\infty k_\infty)(H) = ((\phi_{1,\infty},\phi_{2,\infty})|\alpha|_\infty^{z/2})(b_\infty)\check{S}_+(\mathrm{Ad}(k_\infty)(H))\otimes k_\infty^{-1}.Y^m\overline{X}^n.$$

Our calculation of this factor essentially follows the one in [40] pp.111-113. One first obtains the Iwasawa decomposition

$$\begin{pmatrix}1&0\\1&u\end{pmatrix} = \begin{pmatrix}\frac{u}{\sqrt{1+u\overline{u}}}&\frac{1}{\sqrt{1+u\overline{u}}}\\0&\sqrt{1+u\overline{u}}\end{pmatrix}\begin{pmatrix}\frac{\overline{u}}{\sqrt{1+u\overline{u}}}&-\frac{1}{\sqrt{1+u\overline{u}}}\\\frac{1}{\sqrt{1+u\overline{u}}}&\frac{u}{\sqrt{1+u\overline{u}}}\end{pmatrix}.$$

We therefore get

$$(\phi_\infty|\alpha|_\infty^{z/2})(\begin{pmatrix}\frac{u}{\sqrt{1+u\overline{u}}}&\frac{1}{\sqrt{1+u\overline{u}}}\\0&\sqrt{1+u\overline{u}}\end{pmatrix}) = u^{1-k}\overline{u}^{-n-\ell}|u|_\infty^{z/2}\sqrt{1+u\overline{u}}^{n-m-2-2z}$$

Checking the action of K_∞ on the Lie algebra, we have

$$\check{S}_+(\begin{pmatrix}\frac{\overline{u}}{\sqrt{1+u\overline{u}}}&-\frac{1}{\sqrt{1+u\overline{u}}}\\\frac{1}{\sqrt{1+u\overline{u}}}&\frac{u}{\sqrt{1+u\overline{u}}}\end{pmatrix}.H) = \frac{2\overline{u}}{1+u\overline{u}}.$$

Lastly, we calculate

$$\left\langle Q_{m',n'},k_\infty^{-1}.Y^m\overline{X}^n\right\rangle = (-1)^{m-m'}\overline{u}^{m-m'+n'}\sqrt{1+u\overline{u}}^{-m-n}.$$

Together this gives

$$\left\langle Q_{m',n'},\omega_\infty(\begin{pmatrix}1&0\\1&u\end{pmatrix})(H)\right\rangle = 2(-1)^{m-m'}\frac{(u\overline{u})^{z/2+1}\overline{u}^{m-m'-n+n'}}{u^k\overline{u}^\ell(1+u\overline{u})^{z+2+m}}.$$

This gives rise to Beta-Function integrals, which converge for

$$\mathrm{Re}(z/2) > -(m-m'+1), -(m'+1).$$

The archimedean integral therefore contributes

$$\begin{aligned}(-1)^{m-m'}\frac{i}{4\pi}&\int_{\mathbf{C}^*}\frac{(u\overline{u})^{z/2+1+m-m'}}{(1+u\overline{u})^{z+2+m}}\frac{du\wedge d\overline{u}}{u\overline{u}}\\ =\quad&\frac{(-1)^{m-m'}}{2}\frac{\Gamma(z/2+m-m'+1)\Gamma(z/2+m'+1)}{\Gamma(z+m+2)}.\end{aligned}$$

The preceding analysis shows that all the local integrals converge absolutely for $\mathrm{Re}(z)\gg 0$ and that their product exists so the integral over $\mathbf{A}_F^*$ converges absolutely.

To conclude the proof of the proposition by meromorphic continuation it suffices to prove that for any $\xi\in G(\mathbf{A}_f)$

$$\int_{\sigma_\xi\otimes Q_{m',n'}}\mathrm{Eis}(\Psi^{\mathrm{new},\theta}) = \int_0^\infty \theta_{m',n',\infty}(t)\left\langle Q_{m',n'},\mathrm{Eis}(\begin{pmatrix}1&0\\0&\xi t\end{pmatrix},\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}})(\frac{H}{2})\right\rangle\frac{dt}{t}$$

converges to a meromorphic function in z. The following argument is adapted from [51] Proposition 3.5 and [58] Proposition 2.4.

Recall that $\mathrm{Eis}(\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}}) = \mathrm{Eis}(\omega_z(\phi, \Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}}))$. By picking out the $\frac{\check{H}}{2} \otimes Q^\vee_{m',n'}$-component the integrand equals

$$t^{m-m'+n-n'+k+\ell}\mathrm{Eis}(f(m',n',\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}})(\begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix})),$$

where

$$f(m',n',\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}}) \in V^{K_f^\theta}_{\phi|\alpha|^{z/2}} \otimes M^{m,n}_{\mathbf{C}}$$

is given by

$$(b_\infty k_\infty, g_f) \mapsto \phi_\infty(b_\infty)\tilde{f}(m',n',k_\infty)\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}}(g_f)$$

for a smooth function $\tilde{f}(m',n',\cdot) : K_\infty \to \mathbf{C}^*$.

For $c > 0$ let now

$$I_c(z) := \int_{1/c}^{c} \theta_{m',n',\infty}(t)\mathrm{Eis}(f(m',n',\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}})(\begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix}))\frac{dt}{t}.$$

Since the Eisenstein series has a meromorphic continuation to all $z \in C$ this is a meromorphic function for any $c > 0$. It suffices therefore to show that $I_c(z)$ converges locally uniformly for all z as $c \to \infty$.

Put $E_z(g) = \mathrm{Eis}(f(m',n',\Psi^{\mathrm{new},\theta}_{\phi|\alpha|^{z/2}})(g)$. One checks that the constant term $\mathrm{res}(E_z)(g)$ vanishes for $g = \begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix}$ and $g = \begin{pmatrix}\xi & 0\\ 0 & t\end{pmatrix} w_0$. It follows that

$$I_c(z) = I_c^1(z) + I_c^2(z),$$

where

$$I_c^1(z) = \int_{1/c}^{1} t^{m-m'+n-n'+k+\ell}\left(E_z(\begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix}) - \mathrm{res}(E_z)(\begin{pmatrix}1 & 0\\ 0 & \xi t\end{pmatrix})\right)\frac{dt}{t},$$

$$I_c^2(z) = \int_{1/c}^{1} t^{m'+n'+k+\ell}\left(E_z - \mathrm{res}(E_z)\right)(\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix}\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix} w_0)\frac{dt}{t}.$$

Standard growth estimates for automorphic forms on Siegel sets (see [42] Lemma 3.4, [50] §1.10, [30] I Lemma 10) imply that for any $g \in G(\mathbf{A})$ and $r \in \mathbf{R}$ there exists a constant $C(g,r,z) > 0$, locally uniform in z, such that

$$|E_z(\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} g) - \mathrm{res}(E_z)(\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} g)| \leq C(g,r,z)t^r,\ 0 < t \leq 1.$$

From this it follows that $I_c^1(z)$ and $I_c^2(z)$ converge absolutely and locally uniformly for all z as $c \to \infty$. The limits therefore define meromorphic functions in z, as claimed above.

□

Corollary 19. *For $n-1 < m'+n' < m+1$, $I(\phi,\theta,\Psi^{\mathrm{new}},z)$ converges at $z=0$ and we get $I(\phi,\theta,\Psi^{\mathrm{twist}}_{\phi_f},0) =$*

$$\sum_{[\xi]\in\pi_0(K_f^\theta)} \theta_{m',n'}(\xi)\int_{\sigma_\xi\otimes Q_{m',n'}} \mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f}) =$$

$$= \frac{L(0,\phi_1\theta_{m',n'})L(0,\phi_2^{-1}\theta^{-1}_{m',n'})}{L(0,\chi)} \cdot \frac{\Gamma(m-m'+1)\Gamma(m'+1)}{\Gamma(m+2)} \cdot C(\mathfrak{M}_1, S, \mathfrak{N}),$$

where

$$C(\mathfrak{M}_1, S, \mathfrak{N}) = \frac{(-1)^{n-n'+k+\ell}}{2}(\theta_{m',n'}\phi_2)^{-1}(\mathfrak{M}_1\mathfrak{N}) \cdot \chi^{-1}(\mathfrak{N})\#(\mathcal{O}/\mathfrak{N})^* \cdot \\ \cdot \prod_{v\in S}(\mu_{2,v}^{-1}(-1)\chi^{-1}(\mathfrak{P}_v^{r_v})).$$

□

5. Bounding the denominator

After interpreting the toroidal integral as a cohomological pairing we combine the calculation of Section 4 with results of Hida and Finis to bound the denominator of the Eisenstein cohomology class. For this we need the existence of certain Hecke characters which we construct in Section 5.2.

5.1. **Interpretation of the toroidal integral as evaluation pairing.** Let $F_{\phi,\theta}$ be the finite extension of F_ϕ adjoining the values of the finite part of $\theta_{m',n'}$ and the L-values $L^{\mathrm{alg}}(0, \phi_1\theta_{m',n'})$ and $L^{\mathrm{alg}}(0, \phi_2^{-1}\theta_{m',n'}^{-1})$ and denote its ring of integers by $\mathcal{O}_{\phi,\theta}$. It follows from Proposition 13 that $[\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})] \in H^1(S_{K_f^\theta}, \widetilde{N_{F_{\phi,\theta}}^\vee})$. Let

$$S_{K_f^\theta} \cong \bigoplus_{[\xi]\in\pi_0(K_f^\theta)} \Gamma_\xi^\theta\backslash\mathbf{H}_3$$

for $\{[\xi]\}$ a system of representatives of $\pi_0(K_f^\theta)$. Put

$$\partial_{\{0,\infty\},\xi} := \Gamma_{\xi,B}^\theta\backslash e(B) \cup \Gamma_{\xi,B^{w_0}}^\theta\backslash e(B^w) \subset \Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3$$

and

$$\partial_{\{0,\infty\}} = \bigcup_\xi \partial_{\{0,\infty\},\xi} \subset \partial\overline{S}_{K_f^\theta}.$$

The relative cycles $\sigma_\xi \otimes Q_{m',n'}$ we described in 4.1 give rise to classes in

$$H_1(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3, \partial_{\{0,\infty\},\xi}, j_\xi^*\widetilde{N}_{\mathcal{O}_{\phi,\theta}}).$$

The following Lemma shows that $\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})$ has vanishing constant terms at the ∞- and 0-cusps of each connected component. Since for any automorphic form f the function $f - f_P$ together with its derivatives is fast decreasing at P (see [30] I Lemma 10) this implies that the cocycle gives rise to a differential form fast decreasing at the ∞- and 0-cusps of each connected component. By Proposition 6 the cocycle $\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})$ therefore represents a relative cohomology class in

$$\bigoplus_{[\xi]\in\pi_0(K_f^\theta)} H^1(\Gamma_\xi^\theta\backslash\overline{\mathbf{H}}_3, \partial_{\{0,\infty\},\xi}, j_\xi^*\widetilde{N_{\mathbf{C}}^\vee}),$$

denoted by $[\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}}$, mapping to $[\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})] \in H^1(S_{K_f^\theta}, \widetilde{N_{F_{\phi,\theta}}^\vee})$. This allows us to interpret the toroidal integral of the previous section as sum of evaluation pairings for each connected component so that the value of the integral provides a lower bound on the denominator of $[\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}}$ (see Section 2.6 for properties of the evaluation pairing).

Lemma 20. *We have*

$$[\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}} \in H^1(S_{K_f^\theta}, \partial_{\{0,\infty\}}, \widetilde{N_{F_{\phi,\theta}}^\vee}).$$

Proof. We claim that for $P = B$ and $P = B^{w_0}$

$$\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})_P(g_\infty \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}) = 0$$

for all $g_\infty \in G_\infty$ and all $[\xi] \in \pi_0(K_f^\theta)$. From the form of the constant term for $\mathrm{Eis}(\phi, \Psi_{\phi_f}^0)_B$ (see Proposition 10(a)) we deduce, by interchanging the finite sums of the twists with the integral, that

$$\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})_B = \mathrm{Eis}^\theta(\omega_0(\phi, \Psi_{\phi_f}^{\mathrm{twist}}))_B = \omega_0(\phi, (\Psi_\phi^{\mathrm{twist}})^\theta) + c(\phi, 0)\omega_0(w_0.\phi, (\Psi_{w_0.\phi}^{\mathrm{twist}})^\theta).$$

We need to show that $(\Psi_*^{\mathrm{twist}})^\theta$ vanishes on $\eta_f \begin{pmatrix} 1 & 0 \\ 0 & \xi \end{pmatrix}$ for η the identity matrix and w_0. Then vanishing for η equal to the identity matrix follows immediately from $\sum_{x \in (\mathcal{O}_v/\mathfrak{N}_v)^*} \theta_{m',n',v}^{-1}(x) = 0$ for the finite order character $\theta_{m',n',v}|_{\mathcal{O}_v^*}$ by definition of the conductor $\mathfrak{N}_v$. For $\eta = w_0$ the vanishing follows from our definition of the newvectors Ψ_*^{new}, of which Ψ_*^{twist} is a multiple, and from our choice of $\mathfrak{N}$ coprime to the conductors of the characters ϕ_1 and ϕ_2.

It remains to prove the rationality of $[\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}}$. For this we adapt an argument in [51] Lemma 5.2. Put

$$\omega = [\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})] \text{ and } \omega_{\mathrm{rel}} := [\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}}.$$

Let

$$I_T = \{T_{\pi_v} - \phi_2(\pi_v) - \phi_1(\pi_v)\mathrm{Nm}(\pi_v) : v \notin S \cup T, v \nmid \mathfrak{M}\}.$$

Then I_T annihilates both ω and ω_{rel} (cf. Lemma 9). Proposition 13 implies that $\omega \in H^1(S_{K_f^\theta}, \widetilde{N_{F_{\phi,\theta}}^\vee})$. Thus, using a dimension counting argument, ω is in the image of

$$H^1(S_{K_f^\theta}, \partial_{\{0,\infty\}}, \widetilde{N_{F_{\phi,\theta}}^\vee})[I_T] \to H^1(S_{K_f^\theta}, \widetilde{N_{F_{\phi,\theta}}^\vee})[I_T],$$

where '$[I_T]$' denotes the subspaces annihilated by the elements in I_T. Let c be an element in the left hand side mapping to ω. Then $c - \omega_{\mathrm{rel}} \in H^1(S_{K_f^\theta}, \partial_{\{0,\infty\}}, \widetilde{N_{\mathbf{C}}^\vee})[I_T]$ is in the image of $H^0(\partial_{\{0,\infty\}}, \widetilde{N_{\mathbf{C}}^\vee})$. We recall the description of the boundary cohomology as a $G(\mathbf{A}_f)$-module given by Harder:

$$H^0(\partial \overline{S}_{K_f^\theta}, \widetilde{M_{\mathbf{C}}}) \cong \bigoplus_{\mu: T(\mathbf{Q})\backslash T(\mathbf{A}) \to \mathbf{C}^*} V_{\mu,\mathbf{C}}^{K_f^\theta},$$

where in this case (degree 0) the sum is over characters $\mu = (\mu_1, \mu_2)$ with infinity type (cf. [26] §3.5)

$$\mu_{1,\infty}(z) = z^{-m-k}\overline{z}^{-n-\ell} \text{ and } \mu_{2,\infty}(z) = z^{-k}\overline{z}^{-\ell}.$$

By the Chebotarev density theorem we can find an inert prime q such that $q \equiv 1 \mod \mathfrak{M}_1\mathfrak{M}_2\mathfrak{N}$. We claim that $T_q = [K_f^\theta \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}_q K_f^\theta]$ acts by a scalar factor on $H^0(\partial \overline{S_{K_f^\theta}}, \widetilde{M_{\mathbf{C}}})$. For this consider $\Psi \in V_{\mu,\mathbf{C}}^{K_f^\theta}$ for some μ as above. As in Lemma 9 we get

$$T_q\Psi = (\mu_{2,q}(q) + q^2\mu_{1,q}(q))\Psi.$$

By our assumption on q it therefore acts by $q^{k+\ell} + q^2 \cdot q^{m+n+k+\ell}$, independently of μ. In particular, this also describes the action of T_q on $H^0(\partial_{\{0,\infty\}}, \widetilde{M_{\mathbf{C}}}) \subset$

$H^0(\partial\overline{S_{K_f^\theta}}, \widetilde{M}_{\mathbf{C}})$. Comparing this with the fact that T_q acts on $c-\omega_{\rm rel}$ by $q^{m+k+\ell+1}+q^2\cdot q^{n+k+\ell-1}$ shows that $c-\omega_{\rm rel}=0$. This proves the rationality of $\omega_{\rm rel}$. □

The next lemma shows that the denominator of the relative Eisenstein cohomology class bounds the denominator of the original Eisenstein cohomology class from below.

Lemma 21. *If $p > m \geq n$ then*

$$\delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\rm twist})]) \subseteq \delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\rm twist})]_{\rm rel}).$$

Proof. Put $\omega_{\rm rel} = [\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\rm twist})]_{\rm rel}$ and $\omega = [\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\rm twist})]$. Suppose $a \in \mathcal{O}_{\phi,\theta}$ is such that

$$a\cdot\omega \in H^1(S_{K_f^\theta}, \widetilde{N^\vee_{\mathcal{O}_{\phi,\theta}}})_{\rm free} \text{ but } a\cdot\omega_{\rm rel} \notin H^1(S_{K_f^\theta}, \partial_{\{0,\infty\}}, \widetilde{N^\vee_{\mathcal{O}_{\phi,\theta}}})_{\rm free}.$$

Let λ be a uniformizer of $\mathcal{O}_{\phi,\theta}$ and $m \geq 1$ the smallest integer such that $\lambda^m a\omega_{\rm rel}$ is in the image of an element, say c, of $H^1(S_{K_f^\theta}, \partial_{\{0,\infty\}}, \widetilde{N^\vee_{\mathcal{O}_{\phi,\theta}}})$. Then the image $\overline{c}$ of c in $H^1(S_{K_f^\theta}, \partial_{\{0,\infty\}}, \widetilde{N_k^\vee})$ (where $k=\mathcal{O}_{\phi,\theta}/\lambda$) is nonzero, but its image in $H^1(S_{K_f^\theta}, \widetilde{N_k^\vee})$ is zero. Therefore $\overline{c}$ is in the image of $H^0(\partial_{\{0,\infty\}}, \widetilde{N_k^\vee})$. By Nakayama's Lemma it even has to be in the image of

$$H^0(\partial_{\{0,\infty\}}, \widetilde{N_k^\vee})/H^0(\partial_{\{0,\infty\}}, \widetilde{N^\vee_{\mathcal{O}_{\phi,\theta}}})\otimes k.$$

Note that this quotient is isomorphic to the λ-torsion of $H^1(\partial_{\{0,\infty\}}, \widetilde{N^\vee_{\mathcal{O}_{\phi,\theta}}})$. Under our assumptions p does not divide the level K_f (i.e., $K_v = \mathrm{GL}_2(\mathcal{O}_v)$ for $v \mid p$) so Proposition 2.4.1 (ii) of [54] shows that if $p > \max\{n,3\}$ then $H^1(\partial_{\{0,\infty\}}, \widetilde{N^\vee_{\mathcal{O}_{\phi,\theta}}})$ is torsion-free (the argument in [54] extends to our more general coefficient system). This shows that $\overline{c}=0$, in contradiction to our assumption, so a cannot exist, proving our Lemma. □

5.2. **Construction of special Hecke characters.** Recall that for a Hecke character $\lambda : F^*\backslash\mathbf{A}_F^* \to \mathbf{C}^*$ we defined $\lambda^*(x) = \lambda^{-1}(\overline{x})|x|$. Following constructions by Greenberg [20], Rohrlich [47], and Yang [59] we prove the following:

Lemma 22. (a) *For $F=\mathbf{Q}(\sqrt{-1})$ or $\mathbf{Q}(\sqrt{-3})$ there exists a Hecke character $\mu^{(1,0)}$ of infinity type z with conductor $2\mathcal{D}$ such that $(\mu^{(1,0)})^* = \mu^{(1,0)}$.*

(b) *If $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$ then for any $k>0$ there exists a Hecke character $\mu^{(k,1-k)}$ of infinity type $z^k\overline{z}^{1-k}$ such that $(\mu^{(k,1-k)})^* = \mu^{(k,1-k)}$ whose conductor is given by*

$$\begin{cases} \mathcal{D} & \text{if } d_F \text{ odd,} \\ 2\mathcal{D} & \text{if } d_F \text{ even.} \end{cases}$$

Proof. For $F=\mathbf{Q}(\sqrt{-1})$ and $\mathbf{Q}(\sqrt{-3})$ one can take the inverse of the Grössencharacters associated to the elliptic curves $y^2=x^3+x$ (conductor 64) or $y^2=x^3+1$ (conductor 36), respectively (for curves with minimal conductor divisible only by ramified primes see [20] Lemma p.81). For $F \neq \mathbf{Q}(\sqrt{-1}), \mathbf{Q}(\sqrt{-3})$ we note that Greenberg's construction can be extended to $k \geq 1$: Let $p_1, p_2, \ldots, p_t$ be the rational primes dividing the discriminant d_F and let $\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_t$ be the corresponding primes of F. Since $\mathrm{Nm}(\mathcal{O}_{\mathfrak{p}_i}^*)$ is of index 2 in $\mathbf{Z}_{p_i}^*$ one can define a character of order 2 on $\mathbf{Z}_{p_i}^*$ with kernel containing $\mathrm{Nm}(\mathcal{O}_{\mathfrak{p}_i}^*)$. Via the embedding of $\mathbf{Z}_{p_i}^* \hookrightarrow \mathcal{O}_{\mathfrak{p}_i}^*$

this character can be extended to a character Ψ_i of $\mathcal{O}_{\mathfrak{p}_i}^*$ having finite order (can choose order 2 unless $p_i = 2$ and $4\|d_F$). We can therefore define a continuous homomorphism $\Psi : \mathbf{C}^* \cdot \prod_v \mathcal{O}_v^* \to \mathbf{C}^*$ so that $\Psi(z) = z^k \overline{z}^{1-k}$ for $z \in \mathbf{C}^*$, $\Psi|_{\mathcal{O}_{\mathfrak{p}_i}^*} = \Psi_i$ for $1 \leq i \leq t$, and Ψ is trivial on the other local units. Since -1 is the only non-trivial unit and $\Psi(-1) = 1$ we can define Ψ to be trivial on F^*. This character Ψ can now be extended to a Hecke character $\mu^{(k,1-k)}$ on $\mathbf{A}_F^*$.

We check that $(\mu^{(k,1-k)})^* = \mu^{(k,1-k)}$ by showing that $\mu^{(k,1-k)}|_{\mathbf{A}^*} = \omega_{F/\mathbf{Q}} |\cdot|_{\mathbf{A}^*}$ for $\omega_{F/\mathbf{Q}}$ the quadratic character associated to $F/\mathbf{Q}$ (see [20] for a different proof): Clearly $(\mu^{(k,1-k)}|\cdot|_{\mathbf{A}}^{-1})(t) = 1$ for $t \in \mathbf{R}_{>0}^*$ and $t \in \mathbf{Q}^*$, but

$$(\mu^{(k,1-k)}|\cdot|_{\mathbf{A}}^{-1})(-1) = -1.$$

By construction it is also trivial on $\mathrm{Nm}(\mathbf{A}_F^*)$.

At odd primes the conductor is clearly of index 1. For the calculation of the conductor at the place dividing 2 (and the existence of characters with conductors as claimed) see Rohrlich [47] ($8 \mid d_F$) and Yang [59] ($4\|d_F$). Note that we do not take one of Yang's characters with minimal conductor but one with index 4 at the prime dividing 2. □

Remark 23. (1) We note that any algebraic Hecke character λ satisfying $\lambda^* = \lambda$ is of the form $\mu^{(k,1-k)} \cdot \vartheta$ for a finite order anticyclotomic character ϑ (i.e., such that $\vartheta^c = \overline{\vartheta} = \vartheta^{-1}$) and that they satisfy $\lambda|_{\mathbf{A}^*} = \omega_{F/\mathbf{Q}} |\cdot|_{\mathbf{A}^*}$ with $\omega_{F/\mathbf{Q}}$ the quadratic character of $\mathbf{Q}^* \backslash \mathbf{A}^*$ associated to $F/\mathbf{Q}$.

(2) More generally, for unitary Hecke characters λ satisfying $\lambda^c = \overline{\lambda}$ we have

$$\lambda|_{\mathbf{A}^*} = \begin{cases} 1 & \text{if } \lambda_\infty(-1) = 1, \\ \omega_{F/\mathbf{Q}} & \text{if } \lambda_\infty(-1) = -1. \end{cases}$$

In addition, we note the existence of the following character (cf. [53] Lemme 2.5, [11] Lemma II.1.4(ii)):

Lemma 24. *Let $q \geq 5$ be a rational prime and $\mathfrak{q}$ a prime of F dividing q. Then there exists a Hecke character with conductor $\mathfrak{q}$ of infinity type z.*

Proof. Since $q \geq 5$, $\mathfrak{q}$ separates the roots of unity and so the character is well-defined on $F^* \cdot \mathbf{C}^* U(\mathfrak{q})$, where $U(\mathfrak{q}) := \{x \in \hat{\mathcal{O}}^* | x \equiv 1 \mod \mathfrak{q}\hat{\mathcal{O}}\}$. Since the ray class group $F^* \backslash \mathbf{A}_{F,f}^* / U(\mathfrak{q})$ is finite we can extend trivially to a continuous character on $\mathbf{A}_F^*$. □

5.3. **Bounding the denominator.** Because of Lemma 21 we now assume in addition that $p > m$. We are interested in bounding

$$\delta([\mathrm{Eis}(\Psi_{\phi_f}^0)]) = \{a \in \mathcal{O}_\phi : a \cdot [\mathrm{Eis}(\Psi_{\phi_f}^0)] \in H^1(S_{K_f^S}, \widetilde{(N_{\mathcal{O}_\phi})^\vee})_{\mathrm{free}}\}.$$

Observe that

$$\delta([\mathrm{Eis}(\Psi_{\phi_f}^0)]) \subseteq \delta([\mathrm{Eis}(\Psi_{\phi_f}^{\mathrm{twist}})]) \subseteq \mathcal{O}_\phi,$$

$$\delta([\mathrm{Eis}(\Psi_{\phi_f}^{\mathrm{twist}})])\mathcal{O}_{\phi,\theta} \subseteq \delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]) \subset \mathcal{O}_{\phi,\theta},$$

and (by Lemma 21)

$$\delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]) \subseteq \delta([\mathrm{Eis}^\theta(\Psi_{\phi_f}^{\mathrm{twist}})]_{\mathrm{rel}}).$$

In Section 5.1 we showed that the toroidal integral

$$I(\phi,\theta,\Psi^{\mathrm{twist}}_{\phi_f},0)=\sum_{[\xi]\in\pi_0(K^\theta_f)}\theta_{m',n'}(\xi)\int_{\sigma_\xi\otimes Q_{m',n'}}\mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f})$$

gives the value of a sum of evaluation pairings between relative cohomology and homology. The functoriality of these pairings implies that the denominator of $[\mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f})]_{\mathrm{rel}}$ is bounded below by the denominator of the integral. From Corollary 19 we deduce that

$$I(\phi,\theta,\Psi^{\mathrm{twist}}_{\phi_f},0)=\frac{L^{\mathrm{alg}}(0,\phi_1\theta_{m',n'})L^{\mathrm{alg}}(0,(\phi_2\theta_{m',n'})^{-1})}{L^{\mathrm{alg}}(0,\phi_1/\phi_2)}\cdot C(\mathfrak{M}_1,S,\mathfrak{N}).$$

Since the conductors of ϕ_i and $\theta_{m',n'}$ and $\#(\mathcal{O}/\mathfrak{N})^*$ are coprime to (p) one checks using Lemma 2 that $C(\mathfrak{M}_1,S,\mathfrak{N})\in\mathcal{O}^*_{\phi,\theta}$. This shows that $\delta([\mathrm{Eis}^\theta(\Psi^{\mathrm{twist}}_{\phi_f})]_{\mathrm{rel}})$ is contained in the (possibly fractional) ideal

$$\left(\frac{L^{\mathrm{alg}}(0,\phi_1/\phi_2)}{L^{\mathrm{alg}}(0,\phi_1\theta_{m',n'})L^{\mathrm{alg}}(0,(\phi_2\theta)^{-1}_{m',n'})}\right)\mathcal{O}_{\phi,\theta}.$$

Proposition 25. *If there exists a Hecke character $\theta_{m',n'}$ as in (6) such that $L^{\mathrm{alg}}(0,\phi_1\theta_{m',n'})$ and $L^{\mathrm{alg}}(0,(\phi_2\theta_{m',n'})^{-1})$ lie in $\mathcal{O}^*_{\phi,\theta}$ then*

$$\delta([\mathrm{Eis}(\Psi^0_{\phi_f})])\subseteq L^{\mathrm{alg}}(0,\chi)\mathcal{O}_\phi$$

for $\chi=\phi_1/\phi_2$.

We have at our disposal two results on the non-vanishing modulo p of Hecke L-values as the Hecke character varies in an anticyclotomic $\mathbf{Z}_q$-extension for $q\neq p$:

Theorem 26 ((Finis [14] Thm. 1.1))**.** *Let q be an odd prime split in F, distinct from p. Consider Hecke characters λ of infinity type $\lambda_\infty(z)=z^a\overline{z}^{1-a}$ for a fixed positive integer a with $\lambda^*=\lambda$, conductor dividing dd_Fq^∞ for some fixed d coprime to (p), global root number $W(\lambda)=1$, and such that no inert primes congruent to $-1 \mod p$ divide the conductor of λ with multiplicity one. If $a>1$ then assume p splits in F. Then for all but finitely many such Hecke characters*

$$L^{\mathrm{alg}}(0,\lambda)\ \textit{is a } p-\textit{adic unit.}$$

Hida has proved a similar result:

Theorem 27 (([33] Theorem 4.3))**.** *Assume p splits in F. Fix a character λ of split conductor (i.e., such that the conductor is a product of primes split in $F/\mathbf{Q}$) coprime to p with infinity type $\lambda_\infty(z)=z^a\left(\frac{z}{\overline{z}}\right)^b$ for $a>0$ and $b\geq 0$. Let q be a split prime distinct from p and coprime to the conductor of λ. Then*

$$L^{\mathrm{alg}}(\lambda\vartheta,0)\ \textit{is a } p-\textit{adic unit}$$

for all but finitely many finite-order anticyclotomic characters ϑ of q-power conductor.

Remark. We quoted above the cases of Finis' Theorem when all but finitely many L-values in the anticyclotomic tower are p-adic units; in general this is not true, see [14] for the full statement. Finis also allows ramification at p. Hida's Theorem is actually valid for general CM-fields and also treats the case of non-split q.

We can show then, for example, the following:

Theorem 28. *If $p > m$ is split in F and both ϕ_i have split conductor coprime to (p), then*

$$\delta([\mathrm{Eis}(\Psi^0_{\phi_f})]) \subseteq L^{\mathrm{alg}}(0,\chi)\mathcal{O}_\phi.$$

Proof. By Lemma 24 we can always find a character $\theta_{m',n'}$ of the correct infinity type with split conductor $\mathfrak{N}$ coprime to (pd_F) and the conductors of the ϕ_i such that $\#(\mathcal{O}/\mathfrak{N})^*$ is also coprime to (p). Applying Theorem 27 for both $\phi_1\theta_{m',n'}$ and $\phi_2^{-1}\theta^{-1}_{m',n'}$ there exists a split prime $q \neq p$ coprime to the conductors of the ϕ_i with $q \not\equiv 1 \mod p$ and a finite order anticyclotomic character ϑ of q-power conductor such that $L^{\mathrm{alg}*}(0,\phi_1\theta_{m',n'}\vartheta)$ and $L^{\mathrm{alg}*}(0,(\phi_2\theta_{m',n'}\vartheta)^{-1})$ both lie in $\mathcal{O}^*_{\phi,\theta\vartheta}$ and we can apply Proposition 25 for this modified character $\theta'_{m',n'} = \theta_{m',n'}\vartheta$. □

Remark. This is where our restriction to $m \geq n$ is needed so that the infinity types of $\phi_1\theta_{m',n'}$ and $\phi_2^{-1}\theta^{-1}_{m',n'}$ satisfy the condition of Theorem 27. By using the p-adic functional equation it might be possible to extend Hida's result to $a \leq 1$ and $b \geq 1-a$, which would remove this condition.

For finding congruences between the Eisenstein cohomology class, multiplied by its denominator, and a cuspidal cohomology class, as described in the introduction, we are interested in the case when the restriction of the Eisenstein class to the boundary is integral. As described in Proposition 16 we know that this is the case when $m = n$ and $\chi^c = \overline{\chi}$. The following theorem shows that in this situation there exists (under some conditions on the conductor of χ) an Eisenstein cohomology class with $L^{\mathrm{alg}}(0,\chi)$ as lower bound on the denominator. Note that for $m = n > 0$ the results of Hida and Finis are applicable only for primes p split in F. Recall the definition of the Gauss sum $\tau(\tilde{\chi})$ from Section 2.3.

Theorem 29. *Let χ be a Hecke character of infinity type $z^{m+2}\overline{z}^{-n}$ for $m \geq n \in \mathbf{N}_{\geq 0}$ with conductor $\mathfrak{M}$ coprime to (p). Assume $p > m$ and in addition that either*

(i) p splits in F and χ has split conductor

or

(ii) $m = n$, (if $m > 0$ then also assume that p is split), $\chi^c = \overline{\chi}$, no ramified primes (or 2 if $F = \mathbf{Q}(\sqrt{-3})$) divide $\mathfrak{M}$ and no inert primes congruent to -1 mod p divide $\mathfrak{M}$ with multiplicity one, and

$$\omega_{F/\mathbf{Q}}(\mathfrak{M})\frac{\tau(\tilde{\chi})}{\sqrt{\mathrm{Nm}(\mathfrak{M})}} = 1.$$

Then there exists a character $\phi = (\phi_1,\phi_2)$ with $\chi = \phi_1/\phi_2$ such that the conductor of ϕ_1 is coprime to $(p)\mathfrak{M}$ and

$$\delta([\mathrm{Eis}(\Psi^0_{(\phi_1,\phi_2)_f})]) \subseteq (L^{\mathrm{alg}}(0,\chi)).$$

Proof. Part (i) follows directly from Theorem 28 and Lemma 24. For (ii) we choose $m' = m$ and $n' = 0$ (so that the toroidal integral converges) and $k = 0$, $\ell = -m$. This means that $\theta_{m',n'}$ has to be a finite order character and ϕ_1 should have infinity type z. For suitable ϕ_1 and ϕ_2 we want to apply Theorem 26 to find a finite order anticyclotomic $\theta_{m',n'}$ of q-power conductor, $q \neq p$ split prime coprime to the conductors of the ϕ_i and $q \not\equiv 1 \mod p$, such that both $L^{\mathrm{alg}}(0,\phi_1\theta_{m',n'})$ and $L^{\mathrm{alg}}(0,(\phi_2\theta_{m',n'})^{-1})$ lie in $\mathcal{O}^*_{\phi\theta}$ and such that

$$W(\phi_1\theta_{m',n'}) = W((\phi_2\theta_{m',n'})^{-1}) = 1. \tag{7}$$

The characters ϕ_i must satisfy the conditions on the conductor imposed in Theorem 26 and $\phi_1^* = \phi_1$, $\phi_2^{-1} = (\phi_2^{-1})^*$.

Furthermore, (7) imposes a condition on the root numbers of the ϕ_i as we will now show: Let λ be any Hecke character satisfying $\lambda^* = \lambda$ with conductor $\mathfrak{f}_\lambda$ and ϑ a finite order anticyclotomic character with conductor Q^n for $Q \in \mathbf{Z}$ prime, $Q \neq 2$ and coprime to $\mathfrak{f}_\lambda$. Since $\mathfrak{f}_\lambda = \overline{\mathfrak{f}}_\lambda$ we get $\vartheta(\mathfrak{f}_\lambda) = \pm 1$, but by assumption ϑ has only Q-power roots of unity as values, so $\vartheta(\mathfrak{f}_\lambda) = 1$. Also it is known that $W(\vartheta) = 1$ (see, for example, [19] p. 247 and [16]). By Remark 23 we know $\tilde{\lambda}(Q^n) = \omega_{F/\mathbf{Q}}(Q^n)$ for $\tilde{\lambda} = \lambda/|\lambda|$. Proposition 1 therefore shows that

$$W(\lambda\vartheta) = W(\lambda)W(\vartheta)\tilde{\lambda}(Q^n)\vartheta(\mathfrak{f}_\lambda) = W(\lambda)\omega_{F/\mathbf{Q}}(Q^n). \tag{8}$$

This implies that we need $W(\phi_1) = W(\phi_2^{-1}) = 1$ to be able to satisfy (7) because we are considering $Q = q$ split.

We now define ϕ_1: By possibly twisting $\mu^{(1,0)}$ from Lemma 22 by a finite order anticyclotomic character ϑ with suitable inert conductor we can always ensure by (8) that the resulting character, which we take as ϕ_1, satisfies $\phi_1^* = \phi_1$, $\phi_{1,\infty}(z) = z$, $W(\phi_1) = 1$, and $\mathrm{cond}(\phi_1) = r\mathcal{D}$, for $r \in \mathbf{Z}$ coprime to $p\,\mathfrak{M}$ and such that no inert prime $\equiv -1 \mod p$ divides r with multiplicity one.

One checks that under our assumptions $\chi^c = \overline{\chi}$ and $m = n$ the character $\phi_2^{-1} = \chi/\phi_1$ satisfies $(\phi_2^{-1})^* = \phi_2^{-1}$. From the definition in Section 2.3 we deduce that $W(\chi) = -\frac{\tau(\tilde{\chi})}{\sqrt{\mathrm{Nm}(\mathfrak{M})}}\tilde{\chi}(\mathcal{D}^{-1})$. Now applying Proposition 1 we calculate that

$$W(\phi_2^{-1}) = W(\phi_1^{-1}\chi) = -W(\phi_1^{-1})W(\chi)\omega_{F/\mathbf{Q}}(\mathfrak{M})\tilde{\chi}(r\mathcal{D}) \overset{\text{assumption}}{=} W(\phi_1^{-1}) = W(\phi_1),$$

as desired. Here we use again Remark 23 ($\tilde{\phi}_1|_{\mathbf{A}^*} = \omega_{F/\mathbf{Q}}$ and $\tilde{\chi}|_{\mathbf{A}^*} \equiv 1$), and the last equality holds because $\phi_1^c = \overline{\phi}_1$. By Theorem 26 there exists now some finite order character $\theta_{m',n'}$ such that $L^{\mathrm{alg}}(0, \phi_1\theta_{m',n'})$ and $L^{\mathrm{alg}}(0, (\phi_2\theta_{m',n'})^{-1})$ are simultaneously p-adic units. □

Remark. The condition $\omega_{F/\mathbf{Q}}(\mathfrak{M})\frac{\tau(\tilde{\chi})}{\sqrt{\mathrm{Nm}(\mathfrak{M})}} = 1$ is satisfied, for example, by everywhere unramified characters, so the theorem holds for any split or inert prime p and unramified χ with infinity type z^2.

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