DEFORMATIONS OF SAITO-KUROKAWA TYPE GALOIS REPRESENTATIONS
CASE FOR SUPPORT

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1. Track record

My research area is in algebraic number theory, exploring the connections between modular forms and Galois representations and applications of this to the Bloch-Kato conjecture about special values of L-functions. Establishing the precise links between modular forms (or more generally, automorphic forms) and Galois representations is part of the famous Langlands programme that spans number theory, algebraic geometry and representation theory. My particular focus is the study of Galois representations and automorphic forms over imaginary quadratic fields (aka Bianchi modular forms) and low weight Siegel modular forms, both cases in which previously developed tools from algebraic geometry are not applicable.

I received my PhD in 2005 from the University of Michigan, where I studied under the supervision of Chris Skinner. After postdoc positions at the Max Planck Institute in Bonn and Queens' College, Cambridge, I was appointed to a lectureship in the School of Mathematics and Statistics at the University of Sheffield in 2010. I was awarded an EPSRC first grant (EP/K01174X/1) from 2013-2015, which resulted in 4 papers [BDPŞ15, Ber15, BK16, Ber16] on the relationship between Bianchi and Siegel modular forms, applications to the Bloch-Kato conjecture, p-adic families of modular forms, and the modularity of a particular abelian surface. I have also collaborated with Kris Klosin (CUNY) on projects regarding the modularity of Galois representations, which has resulted in 5 papers to date [BK09, BK11, BK13, BKK14, BK15]. In these papers we develop a new approach to the modularity of residually reducible Galois representations, showing that modularity often follows from congruences between modular forms and instances of the Bloch-Kato conjectures.

The project proposed here is to significantly expand the scope of our method and apply it to prove the first unconditional result about the modularity of abelian surfaces, one dimension up from the case of elliptic curves, for which modularity was proven in the celebrated works of Wiles, Taylor et al. in the 1990s. We will also prove the modularity of a large class of elliptic curves over imaginary quadratic fields, another famous case that has resisted efforts so far. Klosin has recently been awarded an NSA grant to work on these problems. Preliminary work for this project was funded by an LMS research in pairs grant (awarded to the PI) and a Simons Foundation Collaboration grant (awarded to Klosin).

Since 2013 I have given invited talks at international conferences in Bonn, Bulgaria,Copenhagen, Luxembourg, Vienna and Warwick, and the number theory seminars at Bristol, Cambridge, CUNY, Durham, Nottingham, UCL and Warwick. My first PhD student graduated in 2015 and I am currently supervising 2 PhD students. As part of my EPSRC first grant I organized a workshop on Bianchi and Siegel modular forms in Sheffield in 2014 and have been co-organizer for international conferences on automorphic forms and Galois representations at the Banach Centre in Poland in 2011 and 2016.

Regarding the expertise at the host organisation, the University of Sheffield has one of the UK's largest departments of mathematics and research groups in number theory. Following significant investment by the Faculty of Science, several new positions have recently been created. The School of Mathematics and Statistics has a strong research ethos and an excellent group of researchers in algebraic number theory specializing in the Langlands programme, providing the Research Assistant with a stimulating environment and immediate access to experts with complementing expertise. With the "Mathematics and Statistics Research Centre" (MSRC) Sheffield also supports the hosting of short and long term visitors, either as collaborators of members of the School, as visiting lecturers or as workshop participants.

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2. Research Proposal

2.1. **Background.** The key ingredient in Wiles' proof of Fermat's last theorem was to establish the modularity of elliptic curves, 1-dimensional geometric objects given as solutions to equations of the form

$$y^2 = x^3 + ax + b. \tag{1}$$

Modularity here refers to showing that the elliptic curves are related to modular forms, certain holomorphic functions on the complex upper half plane. At the most down to earth level this connection is between the number of solutions of (1) modulo each prime and the coefficients of the Fourier expansion of the modular form, but this can be recast in terms of representations of the absolute Galois group $G_{\mathbf{Q}} = \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. We call an elliptic curve modular if its associated p-adic Galois representation is isomorphic to that associated to a modular form for some prime p. The approach pioneered by Wiles was to consider the residual mod p representation of $G_{\mathbf{Q}}$ corresponding to the p-torsion of the elliptic curve and to show that all p-adic Galois representations reducing to this residual representation are modular. Such deformations of the residual representation are captured by a universal deformation ring R and the modular forms by the Hecke algebra T, and Wiles established an isomorphism between them, a so-called $R = T$ theorem.

The work of Wiles inspired many researchers to work on the Langlands programme, a series of conjectures predicting precise connections between generalisations of modular forms, Galois representations and geometric objects. Despite many impressive advances the analogous question of modularity for abelian varieties of dimension 2 is far from from settled.

The case of such abelian surfaces had first been considered by Yoshida in the 1980's. Instead of "elliptic" modular forms (automorphic forms for GL_2) the relevant automorphic forms are Siegel modular forms for the group GSp_4. In [Yos80] he provided some examples and proved the existence of lifts from elliptic modular forms to Siegel modular forms, proving modularity of special cases of abelian surfaces (comparable somewhat to the case of CM elliptic curves, for which modularity had been proven by Deuring well before Wiles). Tilouine studied the modularity for abelian surfaces in a series of papers [Til06a, Til06b] using p-adic families.

It was not until Brumer and Kramer [BK14] though that the correct level structure was identified and a testable conjecture (the "paramodular conjecture") was formulated. Excluding the cases studied by Yoshida, the conjecture focuses on abelian surfaces whose $\mathbf{Q}$-endomorphisms are just $\mathbf{Z}$. Together with Poor and Yuen they collected numerical evidence for their conjecture. There is also ongoing work of Brumer et al. proving the paramodularity of particular abelian surfaces using a generalisation of the Faltings-Serre method. Cases where the abelian surface acquires additional endomorphisms over $\overline{\mathbf{Q}}$ have been proven by using lifts from Hilbert modular forms over real quadratic fields [JLR12] or Bianchi modular forms (PI in joint work with Dembele, Pacetti and Şengün, [BDPŞ15]).

But so far there are no unconditional results on the paramodularity of abelian surfaces with $\mathrm{End}_{\overline{\mathbf{Q}}}(A) = \mathbf{Z}$. Calegari and Geraghty recently proved results on the modularity of non-regular Galois representations that are residually irreducible and modular. They cover modularity by Siegel modular forms of weight $(k, 2)$, but are not able to treat the parallel weight 2 case needed for abelian surfaces. Pilloni [Pil16] made a significant advance by proving modularity in the weight 2 case, but still assuming that the residual representation is irreducible and modular. The residual modularity assumption in both of these papers is expected to be much harder to prove than Serre's conjecture for GL_2 (where it is a theorem of Khare-Wintenberger). This is due to the fact that for Siegel modular forms weight 2 is too low to be treated by standard cohomological methods, comparable to elliptic modular forms of weight 1.

This project sets out to prove the first general instances of the paramodular conjecture without assuming residual modularity. We will study cases where the abelian surface has a rational torsion point of order a prime $p > 2$. This means that the corresponding p-adic Galois representation is residually reducible. We are particularly interested in the case where the residual representation has 3 irreducible constituents. Residual modularity (for the semi-simplification) is then a consequence of Serre's conjecture for GL_2 and the so-called Saito-Kurokawa lifting from elliptic modular forms to Siegel modular forms. The 4-dimensional Galois representation associated to the Saito-Kurokawa lift of a weight $2k - 2$ elliptic eigencuspform f is reducible and given by $\epsilon^{k-2} \oplus \rho_f \oplus \epsilon^{k-1}$, where ϵ denotes the p-adic cyclotomic character and ρ_f the 2-dimensional p-adic $G_{\mathbf{Q}}$-representation associated to f. We will call residual representations whose semi-simplification agrees with these modulo p "of SK-type".

The residually reducible situation poses major challenges for the study of deformations. The techniques developed to study deformations of residual Galois representations work best in the residually irreducible

case, in particular, the Taylor-Wiles method and its generalizations require this. If the residual representation is “too” reducible there is, in fact, no universal deformation ring. By choosing integral lattices carefully this problem can be overcome, but one then does not expect $R \to T$, but only a map $R^{\rm tr} \to T$ for $R^{\rm tr}$ the subring generated by traces. In the celebrated paper [SW99] Skinner and Wiles managed to circumvent this for GL_2 by using p-adic families and base change techniques. For polarized representations (such as the representations associated to abelian surfaces) Clozel-Harris-Taylor [CHT08] introduced the notion of reducible residual representations of “Schur type", for which $R^{\rm tr}$ and R are closely related. Generalizing the arguments of [SW99] Jack Thorne [Tho15] managed to prove the modularity of such representations. The residual representations of SK type are, however, not of Schur type.
In a series of papers [BK09, BK11, BK13] I developed with Kris Klosin a different method to prove the modularity of polarized residually reducible Galois representations with two Jordan-Hölder factors that are not of Schur type. For this we construct a lattice in the p-adic representation such that the reduction mod p is reducible, but not semisimple. We further identify conditions that ensure that the residual representation is unique up to isomorphism amongst representations of its type (but see [BK15] for results relaxing this assumption). The key ingredient for our method is then a careful study of the reducibility ideal introduced by Bellaïche and Chenevier. We prove the principality of this ideal I and relate the order of R/I to the order of a Bloch-Kato Selmer group. We then establish the modularity of our non-semisimple residual representation by proving the existence of a non-lifted automorphic form that is congruent mod p to the lift corresponding to the semisimple representation. Assuming that one can prove sufficiently many such congruences (namely a lower bound on the congruence module T/J by the order of the Selmer group) this implies the modularity of all reducible deformations. Using the principality of I we can then bootstrap from this to prove the modularity of all deformations. We have recently managed to prove some of the required ingredients to apply our method to the case of 3 irreducible residual constituents for residual representation of SK-type. This allows us to attack the paramodular conjecture for the large class of abelian surfaces with rational torsion without having to assume residual modularity.
Another famous case where the modularity question is open is that of elliptic curves over imaginary quadratic fields $F = \mathbf{Q}(\sqrt{-D})$. As we will explain below this case is related to that of abelian surfaces over $\mathbf{Q}$. By a non-trivial adaption of our method this allows us to prove their modularity in the residually reducible case by automorphic forms for GL_2/F (“Bianchi modular forms”).

2.2. **Research aims and objectives.** I am proposing a major new project to prove the modularity of abelian surfaces (with rational torsion) and of elliptic curves over imaginary quadratic fields, the next major challenges in the Langlands programme linking algebraic geometry and automorphic forms. In particular, I propose to develop new tools in the deformation theory of residually reducible Galois representations and to study the p-adic properties of Saito-Kurokawa lifts. The aim of this project is to advance our understanding of the arithmetic of abelian surfaces and of Siegel modular forms.
My proposed research programme has the following main objectives:

1. construct congruences between Saito-Kurokawa lifts and stable Siegel cuspforms in p-adic families;
2. prove $R = T$ modularity theorems for residually reducible Galois representations of SK-type and prove the paramodular conjecture for abelian surfaces with rational torsion;
3. prove Bianchi modularity of elliptic curves over imaginary quadratic fields with particular torsion structure.

2.3. **National Importance.** Automorphic forms, Galois representations and their L-functions are central to modern number theory and are connected to the practical applications of number theory in cryptography. Their importance is underlined by the fact that two of the Clay Mathematics Million Dollar Millenium Problems deal with L-functions, the Riemann hypothesis and the Birch and Swinnerton-Dyer conjecture. The Langlands programme is a vast research agenda conjecturing precise links between automorphic forms and Galois representations. Three Fields medals have been awarded in the last three decades for work on the Langlands program, and its importance as a central part of pure mathematics is hard to overstate. The International Review of Mathematical Sciences 2010 recommended that number theory in the UK should focus on developing “more analytic and algebraic expertise in all aspects of modular and automorphic forms including the Langlands programme".
Abelian surfaces, the geometric objects we study in our proposal, are of interest not only to geometers and number theorists, but also physicists and cryptographers. One of the main methods of modern data

encryption, introduced by Neal Koblitz in 1989, is based on hyperelliptic curves. The Jacobians of hyperelliptic curves of genus 2 are abelian surfaces, for which proving modularity is one of the key objectives of this proposal.
Research on the modularity of abelian surfaces is carried out by several other groups (including Calegari (Chicago), Pilloni (CNRS/Lyon), Brumer/Voight (New York/Dartmouth), and Dieulefait/ Freitas/ Dembele (Barcelona/Vancouver)). My proposal introduces an innovative and highly promising approach to this problem, which makes this a timely proposal in a very fast moving field.
It is important for UK number theory to build expertise in this area, and keep up with the international developments in the Langlands programme. This research project will also help to cement the position of the University of Sheffield as one of the leading number theory centres in the UK. This proposal complements EPSRC funded research on automorphic forms carried out by Buzzard (Imperial College London), Diamond (Kings' College), and on hyperelliptic curves by T. and V. Dokchitser (Bristol and King's College).

2.4. **Research Programme and Methodology.** In recent joint work with Klosin we established the feasibility of applying our method of proving $R = T$ theorems to the case of residual representations of SK-type. In the level 1 case (representations unramified away from p) we showed how to use Fontaine-Laffaille theory to pin down the residual representation up to isomorphism by constructing a suitable lattice. We also established the principality of the reducibility ideal I of the universal crystalline deformation ring and bounded the number of reducible deformations (measured by the order of the quotient R/I) by the order of a Bloch-Kato Selmer group.
By a result of Fontaine and Abrashkin there are no abelian varieties over $\mathbf{Q}$ with everywhere good reduction. We therefore also started investigating minimal deformations of a ramified residual representation. Amazingly, we showed that uppertriangular deformation can still be related to Selmer groups when one studies minimally ramified deformations whose image at inertia is generated by $\exp(N)$ for nilpotents N of rank 1. This class of representations includes those associated to abelian surfaces with semi-abelian reduction, the main case of interest. By the local Langlands correspondence for GSp_4 (proved by Gan-Takeda) this should correspond exactly to (local representations associated to) Siegel modular forms of squarefree paramodular level.
Our proposal contains 3 intertwined projects. The first project is automorphic and deals with Siegel modular forms and p-adic families thereof. The second studies deformations of Galois representations and establishes the paramodular conjecture for abelian surfaces. The final project proves the Bianchi modularity of elliptic curves over imaginary quadratic fields by using their base change to $\mathbf{Q}$ (and automorphic lifts to Siegel modular forms studied in [BDPŞ15]), which requires extending the results from project 2 to 4 residual pieces.

Project 1: Congruences between Saito-Kurokawa lifts and stable Siegel cuspforms in p-adic families
For the modularity of minimal reducible deformations we need to prove the existence of sufficiently many stable paramodular Siegel cuspforms that are congruent to Saito-Kurokawa lifts. Here "stable" indicates that the corresponding Galois representations are irreducible.
In the level 1 case this was established by Brown [Bro07] for weights $k > 3$, and further extended in [Bro11] and [AB14] to the congruence level $\Gamma_0(N)$ for odd squarefree integers N (and $k > 6$ even). For the lift $SK(f)$ of a classical cuspform f of weight $2k-2$ these congruences modulo a prime $p \nmid N$ are controlled by the p-valuation of the nearly central value $L^{\mathrm{alg}}(f,k)$.
We will prove the analogue of [AB14] for the paramodular level structure (which is different to $\Gamma_0(N)$) and for weights $k \geq 2$. As we are interested specifically in the weight 2 case this is a major project requiring new ingredients, in particular, p-adic interpolation techniques.

Project 2: $R = T$ theorems for residually reducible Galois representations of Saito-Kurokawa type and paramodularity of abelian surfaces
Let A be an abelian surface over $\mathbf{Q}$ of conductor N. Suppose A has a $\mathbf{Q}$-rational point of prime order $p > 2$ and a polarisation of degree prime to p. Then its p-adic Tate module gives rise to a p-adic Galois representation whose reduction has semi-simplification $\bar{\epsilon} \oplus \rho \oplus \mathbf{1}$. When ρ is irreducible then Serre's conjecture tells us that the residual representation $A[p]$ is of SK-type. To prove the paramodularity of A we will therefore study the deformations of residually reducible Galois representations of SK type and generalize our method to prove an $R = T$ theorem in this situation.

Project 3: Bianchi modularity of elliptic curves over imaginary quadratic fields
Let E be an elliptic curve over an imaginary quadratic field K. Despite recent progress by Calegari- Geraghty (see [CG15, Cal16])[1] their expected modularity in terms of so-called Bianchi modular forms is in general unknown, the main obstacle again being the residual modularity. If E has a K-rational torsion point of order p then proving sufficiently many congruences between Eisenstein series and cuspforms would establish the modularity of reducible deformations. The PI studied such cohomological congruences in his PhD thesis [Ber09] and investigated corresponding modularity questions in joint work with Klosin. Because of torsion in the cohomology of Bianchi manifolds it is difficult to prove the modularity of elliptic curves in this manner and p-adic interpolation methods are also difficult to apply.
In this final project we will prove the Bianchi modularity of an elliptic curve E over K by proving instead the paramodularity of the abelian surface A given by the Weil restriction $\mathrm{Res}_{K/\mathbf{Q}}E$. The paramodularity of A implies the Bianchi modularity of E via the (weak) transfer from GSp_4 to GL_4 and a classical result of Arthur and Clozel, which show that the Weil restriction is paramodular by the theta lift of a Bianchi modular form. (The converse direction of deducing paramodularity from Bianchi modularity (established numerically for a particular example) was studied by the PI in [BDPŞ15].) The key advantage of transferring the modularity problem to GSp_4 is that it makes the p-adic interpolation techniques from Project 1(b) available to prove modularity in weight 2 by using results in higher cohomological weights.

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